LEAST-SQUARES ANALYSIS OF DATA
With Unequal Subclass Numbers

ARS H-4
Reprinted February 1975

DATA SYSTEMS
APPLICATION
DIVISION

AGRICULTURAL RESEARCH SERVICE • U.S. DEPARTMENT OF AGRICULTURE

FOREWORD

The analysis of variance is a commonly used statistical method by which estimates of a number of variances are made and by which the significance of the differences between these estimates is determined. More recently the variances have been used as the basis for estimation of variance components. The arithmetic procedures involved are straightforward when the data are based on the same frequency in each subclass.

On the other hand, unequal cell frequencies are not uncommon in some types of experimentation. This is particularly true in animal-breeding experiments concerned with litter size and sex. In addition, animals are frequently lost during the experimental period. As a consequence, the ideal of equal numbers in each subclass or classfication is seldom attained.

The relation of least-squares procedures to the analysis of variance in recent years has been emphasized by many authors. For data conforming to the conventional experimental designs, short-cut procedures are well known which make the least-squares method unnecessary. However, it is only recently that the power of least-squares methods for the analysis of data with unequal subclass frequencies has been recognized.

The method of fitting constants (least-squares analysis) is presented in this article as related to (1) the one-way classification, (2) one-way classification with regression or covariance, (3) two-way classification without interaction, (4) two-way classification with interaction, (5) multiple and nested classifications, and (6) specific animal crossbreeding experiments.

The least-squares methods herein discussed, while related specifically to animal breeding data, are equally applicable to data with unequal subclass frequencies from any other area of research.

Copies of this corrected version may be obtained from Data Systems Application Division (DSAD), Agricultural Research Service, U.S. Department of Agriculture, Room 013, NAL Building, Beltsville, Maryland 20705.

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This publication was formerly published as ARS 20-8 in July 1960 and was reprinted in 1966. This issue has been reviewed and approved for a second reprinting. The current publication reflects the July 1972 reorganization of the Agricultural Research Service.

LEAST-SQUARES ANALYSIS OF DATA WITH
UNEQUAL SUBCLASS NUMBERS

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INTRODUCTION

Disproportionate subclass frequencies always cause the different classes of effects to be non-orthogonal. This means that the different kinds of effects, such as years, sex, and age of dam, cannot be separated directly without entanglement. In order to free these effects from the entanglement or confounding, it is necessary to resort to simultaneous consideration of all effects. With equal subclass frequencies the effects and sums of squares for tests of significance are obtained directly from the class or subclass totals which is equivalent to the simultaneous consideration of all effects. In this case the design is said to be balanced and the effects are all mutually orthogonal.

All statistics which can be computed from data with equal subclass frequencies can also be computed from data with unequal subclass frequencies by use of appropriate statistical methods which are presently available. For example, with fixed effects the investigator is interested in the computation of means, regression coefficients, standard errors, tests of significance, orthogonal comparisons, interactions and mean separation procedures; whereas, with random effects one is interested in the estimation of variance components and tests of significance. Methods of computing these statistics from data with equal subclass frequencies are well known and are given in most standard textbooks for different designs. It is the purpose of this presentation to give presently available procedures for computing all these statistics from data with unequal subclass numbers in considerable detail. The methods are presented by beginning with the simplest models and progressing to the more complex models that are so often encountered in the analysis of animal-breeding data.

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The writer wishes to express appreciation to the professional staff of Biometrical Services for their suggestions in the preparation of the manuscript. Special acknowledgement is given Dr. R. A. Damon, Jr., Biometrician in the Livestock Research Staff, for writing the section concerning the analysis of crossline data and for assistance in the development of several short-cut methods presented.

LEAST-SQUARES PRINCIPLES IN THE ONE-WAY CLASSIFICATION

Methods of analyzing data classified in only one-way are straightforward even though unequal numbers exist from class-to-class. It is intended to demonstrate here that analysis by the method of fitting constants, in which direct matrix arithmetic is involved, yields the same results as the standard methods of computations, thereby clarifying the meaning of least-squares analysis.

Model

In any analysis it is important that the mathematical model underlying the analysis and the assumptions made in using the model be well known. In the one-way classification the model is:

$$y_{ij} = \mu + a_i + e_{ij}$$

$$i = 1, 2, \ldots, p$$

$$j = 1, 2, \ldots, n_i.$$

where:

y_{ij} = the j^{th} observation in the i^{th} A class. (The A classes may represent ages, or years, or sires, or treatments, etc.).

μ = the overall population mean when equal frequencies exist among the classes of A.

a_i = the effect of the i^{th} A class expressed as a deviation from the overall mean μ.

e_{ij} = random errors which are assumed to be independent. If tests of significance are to be made then it is essential to assume that the e_{ij} are also normally distributed, i.e., $N(0,\sigma_e^2)$.

Each of the terms on the right hand side of the equation for the model are population parameters or constants. Estimates of these parameters obtained in the analysis are designated by placing a circumflex (^) over the letter, e.g., $\hat{\mu}$ is the estimate of μ, $\hat{a}_i$ is the estimate of a_i and $\hat{\sigma}_e^2$ is the estimate of σ_e^2. If the a_i are random effects then the variance due to the a_i, σ_a^2, is estimated by $\hat{\sigma}_a^2$.

Least-Squares Equations

The least-squares normal equations result from the use of a differential calculus principle. However, is is unnecessary to actually go through the differential calculus manipulations in order to write out the equations, since they follow a systematic pattern. The first point to remember in the construction of this set of equations is that there must be one equation for each of the constants to be estimated. The equations for the one-way

classification are as follows:

$$
\begin{array}{lllllll}
\mu: & n_{.}\hat{\mu} & + n_1\hat{a}_1 & + n_2\hat{a}_2 & + \ldots & + n_p\hat{a}_p & = Y_{.} \\
a_1: & n_1\hat{\mu} & + n_1\hat{a}_1 & & & & = Y_1 \\
a_2: & n_2\hat{\mu} & & + n_2\hat{a}_2 & & & = Y_2 \\
- & - & & & & & - \\
- & - & & & & & - \\
- & - & & & & & - \\
a_p: & n_p\hat{\mu} & & & & + n_p\hat{a}_p & = Y_p
\end{array}
$$

In this notation, the letter and appropriate subscript on the left followed by a colon denotes the equation. The second subscript is omitted in the n's and in the Y's since this subscript is always summed over. However, a dot in the subscript indicates that summation has been made over that subscript, e.g., $n_{.} = \sum_i n_i$ where n_i = the number of observations in the i^{th} class. The column of values on the right hand side of the equal sign is referred to as the right hand member (RHM). Very often the same variance-covariance matrix is used with several right hand members.

The least-squares equations may be represented in tabular form as follows:

	$\hat{\mu}$	$\hat{a}_1$	$\hat{a}_2$	- - -	$\hat{a}_p$	RHM
μ:	$n_{.}$	n_1	n_2	- - -	n_p	$Y_{.}$
a_1:	n_1	n_1	0	- - -	0	Y_1
a_2:	n_2	0	n_2	- - -	0	Y_2
-	-	-	-	- - -	-	-
-	-	-	-	- - -	-	-
-	-	-	-	- - -	-	-
a_p:	n_p	0	0	- - -	n_p	Y_p

Four features of least-squares equations should be noted from this table. First, the elements of the variance-covariance matrix (the left hand members) form a matrix symmetrical about the main diagonal, i.e., the elements on the right hand side of the diagonal which runs from left to right are a mirror image of those on the left side of these diagonal elements. If the rows and columns are numbered 1, 2, ... p + 1 and the first number is used to identify a given row and the second number to identify a column, then the element in the 12 position is the same as the element in the 21 position, the element in the 23 position is the same as the element in the 32 position, etc. For the problem under consideration, elements in the μ equation, as the coefficients for the $\hat{a}_i$, are the same elements as the diagonal elements in the a_i equations. Thirdly, the off diagonal elements in the section of rows and columns for the a_i are all zero. The fourth feature that should be noted is that the sum of the coefficients for the $\hat{a}_i$ in the μ equation is equal to the coefficient for $\hat{\mu}$ in the same equation. Likewise, the sum of the right

hand members for the a_i equations is equal to the right hand member for the μ equation. All these features are characteristic of original least-squares equations in general, where constants are fitted for classifications.

The least-squares equations for the one-way classification may be shown in a more reduced form as follows:

$$\mu: \quad n_{.} \hat{\mu} + \sum_i n_i \hat{a}_i = Y_{.}$$

$$a_i: \quad n_i \hat{\mu} + n_i \hat{a}_i = Y_i$$

In tabular form, these may be represented as follows:

	$\hat{\mu}$	$\hat{a}_i$	RHM
μ:	$n_{.}$	n_i	$Y_{.}$
a_i:	n_i	$_0\ n_i\ ^0$	Y_i

where the zeros in the $a_i, \hat{a}_i$ section indicate that the off-diagonal elements in this section are equal to zero.

Imposing Restrictions

As noted above, the sum of the coefficients for the $\hat{a}_i$ equals the coefficient for $\hat{\mu}$ in the μ equation and the sum of the RHM's for the a_i equations equals the RHM for the μ equation. Since this is true one cannot solve the equations until some restriction (or constraint) has been imposed on the constants. Actually there are only p independent equations so that an additional relationship must be introduced, generally known as a constraint or restriction, in order to obtain a solution. There are an infinite number of ways in which this can be done from a mathematical viewpoint. However, only two of these are in common use today and they will be considered here.

A direct method of obtaining estimates of the a_i as deviations from $\hat{\mu}$ is to impose the restriction on the least-squares equations that $\sum_i \hat{a}_i = 0$. When this is done the coefficients of one of the $\hat{a}_i$, say $\hat{a}_p$, must be subtracted from the coefficients of the other $\hat{a}_i$. The resulting elements in the a_p equation are then subtracted from the corresponding elements in the other a_i equations to maintain symmetry in the equations. In addition, the RHM for the a_p equation is subtracted from the RHM's for the other a_i equations. When these subtractions are completed the column and row for the a_p equation is deleted and the resulting symmetrical set of equations are solved to obtain estimates of μ and the a_i directly. After subtraction p equations remain, i.e., p-1 for the a_i's and one for μ.

One of the simplest restrictions (or constraints) one could impose on the $\hat{a}_i$ is to assume that one of the $\hat{a}_i = 0$, say $\hat{a}_p$, and simply delete the a_p equation and the column of coefficients for $\hat{a}_p$. When this is done and the

remaining equations are solved to obtain estimates of the unknowns the following linear functions of the parameters are estimated:

$$
\begin{aligned}
\hat{\mu}' &= \mu + \hat{a}_p \\
\hat{a}_1' &= \hat{a}_1 - \hat{a}_p \\
\hat{a}_2' &= \hat{a}_2 - \hat{a}_p \\
&- \\
&- \\
&- \\
\hat{a}_{p-1}' &= \hat{a}_{p-1} - \hat{a}_p \\
\hat{a}_p' &= 0
\end{aligned}
$$

In the model the a_i are expressed as deviations from μ. Hence, it is logical to assume that $\sum_i \hat{a}_i = 0$ and obtain the constant estimates as deviations from $\hat{\mu}$. Of course, the magnitude and sign of the differences among the $\hat{a}_i'$ and among the $\hat{a}_i$ are the same. After obtaining estimates of the a_i' one can compute estimates of the a_i by forcing the sum of the $\hat{a}_i$ to equal zero since $-\sum_{i=1}^{p-1} \hat{a}_i'/p = \hat{a}_p$. The overall mean, $\hat{\mu}$, can then be separated from $\hat{a}_p$ and the least squares means, $\hat{\mu} + \hat{a}_i$, can be computed.

The restriction that the sum of the constant estimates within a given set sum to zero is preferred to the restriction that one of the constants for each set is equal to zero. In addition to the advantage of obtaining the desired constant estimates directly there are other important reasons for this preference that will be explained later.

Utilization of the Matrix Inverse

Numerous methods exist for obtaining the solution of a set of simultaneous equations and for obtaining the inverse of a matrix. None of these will be discussed here. Instead, the use to be made of the inverse elements in analysis of data will be discussed in considerable detail. Some of the uses made of the inverse elements are discussed below.

Computing Estimates of the Constants

Although the constant estimates may be obtained by direct solution of the equations, often it is more convenient to obtain them from the inverse elements and the RHM's of the equations, since

$$\sum_j C^{ij} Y_j = \hat{c}_i$$

where C^{ij} is the inverse element for the i^{th} row and j^{th} column of the complete inverse matrix, Y_j is the RHM for the j^{th} row and $\hat{c}_i$ is the ith

constant estimate. For example,

$$\hat{\mu} = C^{11}Y. + C^{12}(Y_1-Y_p) + C^{13}(Y_2-Y_p) + \dots + C^{1p}(Y_{p-1}-Y_p)$$

$$\hat{a}_1 = C^{21}Y. + C^{22}(Y_1-Y_p) + C^{23}(Y_2-Y_p) + \dots + C^{2p}(Y_{p-1}-Y_p)$$

etc.

If a method of inversion and solution is used so that both the inverse elements and the constant estimates are obtained directly, then an indirect check on the accumulation of rounding errors is available by also computing the constant estimates by this method and checking them against those obtained directly.

Standard Errors of Constant Estimates or Linear Functions of Constant Estimates

The standard error of an estimated constant, such as $\hat{\mu}$ or $\hat{a}_i$, is

$$s_{\hat{c}_i} = \sqrt{C^{ii}\,\hat{\sigma}_e^2}$$

where C^{ii} is the corresponding diagonal inverse element for that constant and $\hat{\sigma}_e^2 = \frac{1}{n.-p}\left[\sum_{ij}\sum y_{ij}^2 - R(\mu,a_i)\right]$ for the one-way classification. The reduction due to fitting all constants, $R(\mu,a_i)$, may be computed from all constant estimates and the original RHM's, i.e.,

$$R(\mu,a_i) = \hat{\mu}\,Y. + \hat{a}_1Y_1 + \hat{a}_2Y_2 + \dots + \hat{a}_pY_p$$

when the restriction is imposed that $\sum_i \hat{a}_i = 0$, since $\hat{a}_p = -\sum_{i=1}^{p-1} \hat{a}_i$. The reduction in sum of squares due to fitting all constants may be computed more easily from the constant estimates and the reduced RHM's, i.e.

$$R(\mu,a_i) = \hat{\mu}\,Y. + \hat{a}_1(Y_1-Y_p) + \dots + \hat{a}_{p-1}(Y_{p-1}-Y_p).$$

The reduction in sum of squares due to fitting all constants is the same when the a_p equation is deleted by row and column, i.e.,

$$R(\mu,a_i) = \hat{\mu}'Y. + \hat{a}_1'Y_1 + \hat{a}_2'Y_2 + \dots + \hat{a}_{p-1}'\,Y_{p-1}$$

However, when interactions are included in the model difficulties arise in imposing the restriction that certain effects equal zero. These difficulties are discussed in the section on the two-way classification with interaction.

The standard error of a difference between two constant estimates is

$$s_{\hat{a}_i-\hat{a}_j} = \sqrt{(C^{ii} + C^{jj} - 2C^{ij})\,\hat{\sigma}_e^2}$$

The same standard error for a difference between two constants is obtained regardless of the restriction imposed on the constant estimates. When the restriction was imposed that $\hat{a}_p = 0$, the inverse elements for the a_p column and row all equal zero, but when the restriction was imposed that $\sum_i \hat{a}_i = 0$ the inverse elements for the a_p-column and-row are computed from the dependencies which cause the inverse elements within set to sum to zero by rows and columns. For example,

$$c^{1(p+1)} = -(c^{12} + c^{13} + c^{14} + \ldots + c^{1p})$$

$$c^{2(p+1)} = -(c^{22} + c^{23} + c^{24} + \ldots + c^{2p})$$

$$\vdots$$

$$c^{(p+1)(p+1)} = -(c^{1(p+1)} + c^{2(p+1)} + c^{3(p+1)} + \ldots + c^{p(p+1)})$$

The standard errors of the least-squares means are obtained as follows:

$$s_{\hat{\mu}+\hat{a}_i} = \sqrt{(c^{11} + c^{ii} + 2c^{1i})\,\hat{\sigma}^2_e}$$

Regardless of the restriction imposed, this standard error will be the same, but only in the case of the one-way classification. With multiple classifications, this standard error is difficult to obtain if a column and row are deleted in each set of constants. With the restriction that the sum of the constant estimates sum to zero within each set, the least-squares means and their standard errors are easy to compute from the constants and the inverse elements.

Computing Sums of Squares for the Analysis of Variance

The sum of squares for A is easily obtained directly from the class numbers and class totals in the one-way classification. However, this same sum of squares may be obtained in the more general way as shown below:

$$\text{S.Sqs.} = B'\, Z^{-1}\, B$$

where B' is a row vector of the constant estimates for a given set (such as the $\hat{a}_i$); Z^{-1} is the inverse of the segment of the inverse of the variance-covariance matrix corresponding, by row and column, to this set of constants; and B is a column vector of the set of constants. The sum of squares obtained in this manner is equal to the reduction in sum of squares due to fitting all constants minus the reduction in sum of squares due to fitting all constants except the set being considered. This sum of squares may be obtained in this manner regardless of which of the two restrictions have been imposed on the constants. Obviously, the addition of the extra row and column is unnecessary if $\hat{a}_p$ has been assumed to equal zero. The last row and column is also unnecessary if the restriction was imposed that $\sum_i \hat{a}_i = 0$.

Numerical Example

Suppose the data given in the table below represent gains of individual barrows that were randomly assigned to three different rations.

	Ration No.		
Pig No.	1	2	3
1	3	5	7
2	5	6	6
3	6	2	4
4	2	7	3
5		8	6
6		3	4
7		9	
8		8	
Y_i	16	48	30
$\bar{y}_i$	4	6	5

Completing the analysis of these data by least-squares procedures requires the following steps:

1) Mathematic model:

$$y_{ij} = \mu + r_i + e_{ij}$$

where:

y_{ij} = the gain of the j^{th} barrow on the i^{th} ration.

μ = overall mean with equal numbers.

r_i = effect of the i^{th} ration.

e_{ij} = random error assumed to be independent and normally distributed, i.e., $NID(0,\sigma_e^2)$.

2) Least-squares equations:

	$\hat{\mu}$	$\hat{r}_1$	$\hat{r}_2$	$\hat{r}_3$	RHM
μ:	18	4	8	6	94
r_1:	4	4	0	0	16
r_2:	8	0	8	0	48
r_3:	6	0	0	6	30

3) Imposing the restriction that $\sum_i \hat{r}_i = 0$:

The reduced least-squares equations are as follows:

	$\hat{\mu}$	$\hat{r}_1$	$\hat{r}_2$	RHM
μ:	18	-2	2	94
r_1:	-2	10	6	-14
r_2:	2	6	14	18

4) Inversion of the variance-covariance matrix and computation of the constants:

$$\begin{pmatrix} 18 & -2 & 2 \\ -2 & 10 & 6 \\ 2 & 6 & 14 \end{pmatrix}^{-1} = \begin{pmatrix} .060185 & .023148 & -.018519 \\ \underline{.023148} & .143519 & -.064815 \\ -.018519 & -.\underline{064815} & \underline{.101852} \end{pmatrix} \quad \ldots\ldots \quad (1)$$

$$\hat{\mu} = (.060185)(94) + (.023148)(-14) - (.018519)(18) = 5$$
$$\hat{r}_1 = (.023148)(94) + (.143519)(-14) - (.064815)(18) = -1$$
$$\hat{r}_2 = (-.018519)(94) - (.064815)(-14) + (.101852)(18) = 1$$
$$\hat{r}_3 = -(\hat{r}_1 + \hat{r}_2) = 0.$$

5) Setting up the analysis of variance:

$$\begin{aligned} \text{Error S. Sqs} &= \sum_i\sum_j y_{ij}^2 - R(\mu, r_i) \\ &= 568 - (5)(94) - (-1)(-14) - (1)(18) \\ &= 568 - 502 = 66. \end{aligned}$$

$$\begin{aligned} \text{Ration S.Sqs.} &= B' Z^{-1} B \\ &= \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} .143519 & -.064815 \\ -.\underline{064815} & \underline{.101852} \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} \underline{9.7778} & 6.2222 \\ 6.2222 & \underline{13.7778} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -3.5556 & 7.5556 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= 11.1112. \end{aligned}$$

The analysis of variance is as follows:

Source of Variation	d.f.	S.Sqs.	M.S.	F
Rations	2	11.1112	5.5556	1.263 n.s.
Error	15	66.0000	4.4000	

It is easily verified that the sum of squares for rations, 11.1112, is equal to

$$\frac{(16)^2}{4} + \frac{(48)^2}{8} + \frac{(30)^2}{6} - \frac{(94)^2}{18}$$

and that the error sum of squares is equal to

$$\sum_{i}\sum_{j} y_{ij}^2 - \sum_{i} \frac{Y_{i.}^2}{n_{i.}} = 568 - 502 = 66$$

Hence, in the one-way classification the between group uncorrected sum of squares, $\sum_{i} \frac{Y_{i.}^2}{n_{i.}}$ equals $R(\mu, a_i)$, even though the number of observations varies from group-to-group. Also, in the one-way classification $\hat{\mu}$ and the $\hat{r}_i$ can be obtained directly from the class or group means as follows:

$$\hat{\mu} = \frac{1}{p}\left(\frac{Y_1}{n_1} + \frac{Y_2}{n_2} + \dots + \frac{Y_p}{n_2}\right) = \frac{1}{p}\sum_{i} \bar{y}_i$$

$$= \frac{1}{3}(4 + 6 + 5) = 5 \text{ in this example, and}$$

$$\hat{r}_i = \bar{y}_i - \hat{\mu}.$$

Hence, in the present example,

$$\hat{r}_1 = 4 - 5 = -1$$

$$\hat{r}_2 = 6 - 5 = 1, \text{ and}$$

$$\hat{r}_3 = 5 - 5 = 0 .$$

6) Standard errors and individual comparisons:

(a) The standard error of an effect, such as the $\hat{r}_i$, has little significance itself since it cannot exist without the mean. The standard errors will be computed for the least-squares means, $\hat{\mu}$ and the $\hat{\mu} + \hat{r}_i$.

$$s_{\hat{\mu}} = \sqrt{(.060185)(4.4)} = .51$$

$$s_{\hat{\mu}+\hat{r}_1} = \sqrt{[.060185 + .143519 + (2)(.023148)](4.4)}$$

$$= \sqrt{(.25)(4.4)}$$

$$= 1.05$$

$$s_{\hat{\mu}+\hat{r}_2} = \sqrt{[.060185 + .101852 + (2)(-.018519)](4.4)}$$

$$= \sqrt{(.125)(4.4)}$$

$$= .74$$

$$s_{\hat{\mu}+\hat{r}_3} = \sqrt{[.060185 + .115741 + (2)(-.004629)](4.4)}$$

$$= \sqrt{(.166668)(4.4)}$$

$$= .86$$

The inverse elements for the coefficients in the least-squares equation that were subtracted out above are necessary to obtain the standard error for $\hat{\mu} + \hat{r}_3$. These were obtained as follows:

$$.115741 = .143519 + .101852 + (2)(-.064815)$$

and $$-.004629 = -(.023148 - .018519).$$

Of course, in the one-way classification it is unnecessary to go through all this manipulation to obtain these standard errors. They may be obtained directly from the well known formula,

$$s_{\bar{y}_i} = \sqrt{\frac{s^2}{n}}$$

where s^2 is the estimate of the error variance, $\hat{\sigma}_e^2$, and n is the number in the i^{th} group. In order to show that the two methods give identical results, they are computed directly below:

$$s_{\hat{\mu}+\hat{r}_1} = \sqrt{\frac{4.4}{4}} = 1.05$$

$$s_{\hat{\mu}+\hat{r}_2} = \sqrt{\frac{4.4}{8}} = .74$$

$$s_{\hat{\mu}+\hat{r}_3} = \sqrt{\frac{4.4}{6}} = .86$$

(b) Individual comparisons among the $\hat{r}_i$.

The well known "t" test can be employed to compare any two of the $\hat{r}_i$ effects if desired, but becomes less satisfactory as the number of effects increase. The ratio, distributed as t, is

$$t = \frac{\hat{r}_i - \hat{r}_j}{s_{\hat{r}_i - \hat{r}_j}} .$$

Tests of significance among all pairs of the $\hat{r}_i$ in this example using the "t" test are given below:

(i) $\hat{r}_1 - \hat{r}_2 = -2$

$$t = \frac{-2}{\sqrt{[.143519 + .101852 - (2)(-.064815)]\,(4.4)}}$$

$$= \frac{-2}{\sqrt{(.375)(4.4)}}$$

$$= \frac{-2}{1.285}$$

$$= -1.56$$

$$\text{(ii)}\quad \hat{r}_1 - \hat{r}_3 = -1$$

$$t = \frac{-1}{\sqrt{[.143519 + .115741 - (2)(-.078704)]\,(4.4)}}$$

$$= \frac{-1}{\sqrt{(.416668)(4.4)}}$$

$$= \frac{-1}{1.354}$$

$$= -.74$$

$$\text{(iii)}\quad \hat{r}_2 - \hat{r}_3 = 1$$

$$t = \frac{1}{\sqrt{[.101852 + .115741 - (2)(-.037037)]\,(4.4)}}$$

$$= \frac{1}{\sqrt{(.291667)(4.4)}}$$

$$= \frac{1}{1.133}$$

$$= .88$$

Duncan's multiple range test, as modified by Kramer,[2] may be employed to make all pairwise comparisons with the use of the inverse elements and the standard deviation for error. If the values

$$(\bar{y}_i - \bar{y}_j)\sqrt{\frac{2}{C^{ii} + C^{jj} - 2C^{ij}}}$$

are greater than $\hat{\sigma}_e\, z_{p,n_2}$ the difference is significant; where z_{p,n_2} is the significant studentized range value in Duncan's tables (either the .05 or .01 table), p is the number of means in the range chosen

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[2] Kramer, Clyde Young. Extension of multiple range tests to group correlated adjusted means. Biometrics 13: 13-18. 1957.

and n_2 is the number of degrees of freedom for error. For computational convenience and for this example these tests are made in tabular form below at the .05 level of probability:

Comparison	$\bar{y}_i - \bar{y}_j$	$\sqrt{\frac{2}{c^{ii}+c^{jj}-2c^{ij}}}$	Product Differences	$\hat{\sigma}_e \, z_{p,n_2}$
$\hat{r}_2$ vs $\hat{r}_3$	1	2.619	2.62	6.31
$\hat{r}_2$ vs $\hat{r}_1$	2	2.309	4.62	6.63
$\hat{r}_3$ vs $\hat{r}_1$	1	2.191	2.19	6.31

All differences, of course, are non-significant for this example. These tests are given merely to illustrate the computational procedures involved.

The sums of squares required to perform tests of significance for orthogonal contrasts among a set of constants are obtained by transforming the segment of the inverse corresponding to the set of constants. This transformation is equivalent to the computation of the inverse elements corresponding to the variances and covariances of the selected orthogonal contrasts. These elements are most easily computed by means of a relatively simple matrix multiplication procedure. If T is the transformed segment of the inverse, K the transformation matrix and Z_A the square segment of the variance-covariance inverse with the additional column and row added it can be shown that $T = KZ_AK'$.

The transformation matrix K is obtained from the orthogonal coefficients which define the desired set of orthogonal contrasts. Suppose the following orthogonal comparisons are desired among the three rations in the present example:

$\hat{r}_1$	$\hat{r}_2$	$\hat{r}_3$
2	-1	-1
0	1	-1

Then the transformation matrix is:

$$K = \frac{1}{4}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 2 & -2 \end{bmatrix}$$

It should be noted that the sum of the coefficients by row in the transformation matrix sum to zero by row with the sign considered but to unity when disregarding the sign.

The transformation of the complete segment of inverse elements for the $\hat{r}_i$ and the computation of the sums of squares for each contrast are given below:

$$KZ_A = \frac{1}{4}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} .143519 & -.064815 & -.078704 \\ -.064815 & .101852 & -.037037 \\ -.078704 & -.037037 & .115741 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} .430557 & -.194445 & -.236112 \\ .027778 & .277778 & -.305556 \end{bmatrix}$$

$$T = KZ_AK' = \frac{1}{4}\begin{bmatrix} .430557 & -.194445 & -.236112 \\ .027778 & .277778 & -.305556 \end{bmatrix} \frac{1}{4}\begin{bmatrix} 2 & 0 \\ -1 & 2 \\ -1 & -2 \end{bmatrix}$$

$$= \frac{1}{16}\begin{bmatrix} 1.291671 & .083334 \\ .083334 & 1.166668 \end{bmatrix}$$

An easier method of computing T is bv subtracting the coefficients in the last column of the transformation matrix K from the coefficients in the other columns prior to the matrix multiplications. In this case, the last column and row for Z need not be added, i.e.,

$$T = K_AZK_A' = KZ_AK'$$

The constants for the two orthogonal contrasts are obtained from the transformation matrix and the $\hat{r}_i$ constants, as follows:

$$\frac{1}{4}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \end{bmatrix}$$

Therefore, $\hat{c}_1 = -\frac{3}{4}$ and $\hat{c}_2 = \frac{1}{2}$ where $\hat{c}_i$ is the estimate of the constant for the contrast. It will be noted that these constants are one-half of the difference obtained for the contrast from the $\hat{r}_i$ constants directlv and the orthogonal coefficients.

The sums of squares for the two orthogonal single degree of freedom contrasts are now obtained from $B'Z^{-1}B$ in the usual manner.

S.Sqs. for Ration 1 vs. Rations 2 and 3

$$= \frac{(-3/4)^2\ (16)}{1.291671} = 6.9677$$

S.Sqs. for Ration 2 vs. Ration 3

$$= \frac{(1/2)^2\ (16)}{1.166668} = 3.4286$$

The sums of squares for this set of orthogonal contrasts do not add up to the sum of squares among the rations shown in the analysis of variance table on page 9, i.e., 6.9677 + 3.4286 ≠ 11.1112. This is true because of unequal numbers among rations. In effect, the partitioning of the two degrees of freedom among the rations into the chosen set of orthogonal comparisons has created a two-way

classification situation with unequal subclass numbers. If the same means are compared with weighted coefficients that are directly proportional to the unequal frequencies the sums of squares for the contrasts will add to the sum of squares for rations.

7) A short-cut method of computing the inverse of the variance-covariance matrix and the constant estimates:

When constants are fitted for all degrees of freedom among a set of classes or subclasses the inverse of the variance-covariance matrix and the constant estimates can often be obtained most easily with the aid of a transformation matrix. If D is the diagonal least-squares coefficient matrix for the classes or subclasses and K is the transformation matrix, then the inverse of the variance-covariance matrix (C) is computed from $KD^{-1}K'$. The constant estimates are obtained from KS = B, where S is a column vector of the class or subclass means and B is a column vector of the constants.

The diagonal least-squares coefficient matrix (D) for the present one-way classification example is

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 6 \end{bmatrix} .$$

These are the left hand members for the least-squares equations for the $\mu + r_i$. The transformation matrix is

$$K = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ -1 & 2 & -1 \end{bmatrix} ,$$

since this is the functional relationship of $\mu + r_i$, the ration means, and the two kinds of constants μ and r_i, i.e.,

$$\hat{\mu} = \frac{\hat{s}_1 + \hat{s}_2 + \hat{s}_3}{3}$$

$$\hat{r}_1 = \hat{s}_1 - \frac{\hat{s}_1 + \hat{s}_2 + \hat{s}_3}{3}$$

$$= \frac{2}{3}\hat{s}_1 - \frac{1}{3}\hat{s}_2 - \frac{1}{3}\hat{s}_2 ,$$

$$\hat{r}_2 = -\frac{1}{3}\hat{s}_1 + \frac{2}{3}\hat{s}_2 - \frac{1}{3}\hat{s}_3 .$$

where the $\hat{s}_i$ are the subclass (ration) means.

Since D is a diagonal matrix, its inverse consists only of reciprocals of the diagonal elements. Hence, the inverse of the variance-covariance matrix and the constant estimates can be computed individually

by setting up the following table:

	μ	r_1	r_2	Ration Means	D^{-1}	Constants
s_1	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	4	.250000	5
s_2	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	6	.125000	-1
s_3	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	5	.166667	1

The constants in the last column of the above table are computed by multiplying each of the columns in the transpose of the transformation matrix on the left in turn with the column of ration means. For example,

$$\hat{\mu} = \frac{1}{3}(4 + 6 + 5) = 5$$

$$\hat{r}_1 = \frac{1}{3}\left[(2)(4) - 6 - 5\right] = -1$$

$$\hat{r}_2 = \frac{1}{3}\left[-4 + (2)(6) - 5\right] = 1 .$$

Since $\hat{r}_1 + \hat{r}_2 + \hat{r}_3 = 0$, then $\hat{r}_3$ can also be easily computed.

The inverse elements of the variance-covariance matrix are obtained by multiplying each column of the transpose of the transformation matrix by itself and with each of the other columns in turn and then multiplying this product by the D^{-1} column. For example,

$$c^{11} = \frac{1}{9}(.250000 + .125000 + .166667)$$

$$= .060185,$$

$$c^{12} = \frac{1}{9}\left[(2)(.250000) - .125000 - .166667\right]$$

$$= .023148,$$

etc.

The inverse, obtained in this manner, is presented below:

$$C = \begin{bmatrix} \underline{.060185} & .023148 & -.018519 \\ .023148 & \underline{.143519} & -.064815 \\ -.018519 & -.064815 & \underline{.101852} \end{bmatrix}$$

It will be noted that these elements agree with the inverse given in (1). The inverse computed in this manner can be obtained as accurately as desired since the only source of error is the number of significant digits carried in

the reciprocals of the diagonal elements. The same is true for the constants and the class or subclass means.

Summary of the One-way Classification Analysis

Although in practice one would not use the least-squares procedures when the data are classified in only one way, it is evident that the same results can be obtained with these procedures as with commonly used direct calculation procedures. In addition, the principles and computational procedures of least-squares analysis can be more easily introduced by way of the one-way classification analysis. Extension of these procedures from the one-way classification to more complex designs is straight-forward for one who is familiar with matrix arithmetic. The extensions to the more complex designs are given in later sections. Also some additional least-squares computational procedures are introduced in the remaining sections which are not applicable to the one-way classification. These are the absorption of a set of equations and the computation of coefficients for variance components in the expectation of adjusted sums of squares for effects that have been absorbed. Procedures for obtaining coefficients for variance components in least-squares sums of squares or cross-products are given in the section for the two-way classification.

ONE-WAY CLASSIFICATION WITH REGRESSION OR COVARIANCE

Covariance analysis of data from a one-way classification can be accomplished by standard methods even though unequal numbers exist from class-to-class. However, identical results can be obtained with least-squares procedures, which directly involve matrix arithmetic. These procedures are applicable to a wide range of problems. If mean separation procedures are to be used on the least-squares class means or if a set of orthogonal comparisons is desired, it is best to obtain the inverse of the variance-covariance matrix. An alternative procedure for obtaining this inverse, which is often useful, is presented in this section for the one-way classification, when a continuous[3/] independent variable must also be considered. The method is applicable also to more complex problems.

Model

The mathematical model for the one-way classification with a regression may be written in two ways. If the continuous independent variable (X) is to be taken as a deviation from the mean (of X), then the model is as follows:

3/ Actually, it is not necessary for the covariate to be a continuous variable. However, except for the case of only two classes such as sex, it is usually more satisfactory to fit individual constants for discrete classifications rather than regressions. Hence, the covariate will always be referred to here as a continuous variate rather than discrete.

$$y_{ij} = \mu + a_i + b(X_{ij} - \bar{x}) + e_{ij}$$

$$i = 1, 2, ---, p$$

$$j = 1, 2, ---, n_i$$

where: y_{ij} = the j^{th} observation in the i^{th} A class,

μ = the overall mean for the y_{ij} when equal frequencies exist in each of the A classes,

a_i = the effect of the i^{th} A class,

b = partial regression of the dependent variable (y) on the independent continuous variable (X) holding the discrete variable (the a_i) constant. The discrete variable or variables are held constant in a covariance analysis by estimating b on an error line basis. In the one-way classification this means that b is estimated on an intra-group basis.

X_{ij} = the continuous independent variate for the corresponding y_{ij} observation. The X_{ij} are regarded as fixed and measured without error,

$\bar{x}$ = the arithmetic mean of the X_{ij},

e_{ij} = the random errors. These are assumed to be independent and if tests of significance are to be made they also must be assumed to be normally distributed.

In practice, it is more convenient to work with the values of X_{ij} rather than the plus and minus deviations of the X_{ij} from the mean, $\bar{x}$. When this is done the descriptive working model is as follows:

$$y_{ij} = \alpha + a_i + bX_{ij} + e_{ij} ,$$

where the definition of all common terms remains the same. The new symbol, α, is the population mean when X is equal to zero. Using this model it is therefore necessary to obtain the estimate of μ, from

$$\hat{\mu} = \hat{\alpha} + \hat{b}\,\bar{x}$$

<u>Least-Squares Equations</u>

The least-squares equations for the one-way classification with one continuous independent variable are shown below in tabular form:

	$\hat{\alpha}$	$\hat{a}_i$	$\hat{b}$	RHM
α:	$n.$	n_i	$X.$	$Y.$
a_i:	n_i	$0 \;\; n_i \;\; 0$	X_i	Y_i
b:	$X.$	X_i	$\sum_{ij}\sum X^2_{ij}$	$\sum_{ij}\sum X_{ij}y_{ij}$

If the X_{ij} are expressed as deviations from $\bar{x}$ then the equation for b becomes

$$\sum_{ij}\sum(X_{ij} - \bar{x})\,\hat{a}_i + \sum_{ij}\sum(X_{ij} - \bar{x})^2\hat{b} = \sum_{ij}\sum(X_{ij} - \bar{x})y_{ij}$$

and μ is estimated directly rather than α.

Imposing Restrictions

The restrictions imposed in order to obtain a unique solution to these equations are the same as for the one-way classification without the regression being included. Thus, no restriction is necessary for the regression equation. In the subtraction procedure when the restriction of $\sum_i \hat{a}_i = 0$ is imposed the column and row coefficients for the b equation are treated in the same manner as the RHM values.

Inversion of the Reduced Matrix and Solution of the Equations

When constants for all the degrees of freedom among a set of classes or subclasses are being fitted, in addition to the regression (or regressions), an alternative procedure is available for obtaining the inverse of the variance-covariance matrix. In this case the regression(s) are being fitted on a "within" class or subclass basis and can be calculated from the equation(s) obtained on a "within" basis. For example, in the simple case being considered here the least-squares estimate of b is given by

$$\hat{b} = \frac{\sum_{ij}\sum X_{ij}y_{ij} - \sum_i \frac{X_i Y_i}{n_i}}{\sum_{ij}\sum X^2_{ij} - \sum_i \frac{X_i^2}{n_i}}$$

The diagonal inverse element of the variance-covariance matrix corresponding to the b is

$$\frac{1}{\sum_{ij}\sum X^2_{ij} - \sum_i \frac{X_i^2}{n_i}} \, .$$

The problem is to adjust the inverse elements of the diagonal coefficient

matrix of the $\alpha + a_i$ and then transform this adjusted inverse matrix with the appropriate transformation matrix. In order to obtain the inverse elements of the $\alpha + a_i$ section it is first necessary to obtain the inverse elements of the variance-covariance matrix for the b-column or-row. These elements are referred to as the G section of the variance-covariance inverse and are obtained as follows:

$$G^{ib} = G^{bi} = -\frac{X_i}{n_i\left[\sum_{ij}\sum X_{ij}^2 - \sum_i \frac{X_i^2}{n_i}\right]} .$$

The diagonal inverse elements for the $\alpha + a_i$ section, A, are then computed from

$$A^{ii} = \frac{1}{n_i}(1 - X_i G^{bi})$$

and the off-diagonal inverse elements for the $\alpha + a_i$ section are given by

$$A^{ij} = -\frac{1}{n_i}(X_i \; G^{bj})$$

or

$$A^{ij} = -\frac{1}{n_j}(X_j G^{bi}) \quad .$$

The inverse of the variance-covariance matrix for the α and a_i separately may then be obtained from the matrix multiplication KAK', where K is the functional relationship matrix between the classes (the $\alpha + a_i$) and the individual constants (α and a_i), and A is the inverse matrix for the $\alpha + a_i$. In the one-way classification, as was shown in the numerical example in the previous section on the one-way classification,

$$K = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \frac{p-1}{p} & -\frac{1}{p} & -\frac{1}{p} & \cdots & -\frac{1}{p} \\ -\frac{1}{p} & \frac{p-1}{p} & -\frac{1}{p} & \cdots & -\frac{1}{p} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -\frac{1}{p} & -\frac{1}{p} & -\frac{1}{p} & \cdots \frac{p-1}{p} & -\frac{1}{p} \end{bmatrix}$$

The estimates of the constants, $\hat{\alpha}$ and the $\hat{a}_i$, may then be obtained in either of two ways. The inverse matrix for the $\alpha + a_i$ and b may be multiplied by the appropriate right hand members, i.e. the $Y_{i.}$ and $\sum_{ij}\sum X_{ij}y_{ij}$, one row (or column) at a time, to obtain the $\hat{\alpha} + \hat{a}_i$. Then by imposing the restriction that $\sum_i \hat{a}_i = 0$, the estimate of α is obtained from

$$\hat{\alpha} = \frac{\sum_i(\hat{\alpha} + \hat{a}_i)}{p} \quad .$$

and the $\hat{a}_i$ are then computed.

The second method of computing $\hat{\alpha}$ and the $\hat{a}_i$ is from the complete variance-covariance inverse matrix for α, a_i, and b and also the right hand members for the reduced least squares matrix. The inverse elements obtained above (the G's) for the b-row and-column must be transformed so that they will correspond to the variance-covariance matrix for α, a_i, and b rather than to $\alpha + a_i$, and b. These are easily obtained from KG, where G is the column vector of the G^{ib}.

Computation of Sums of Squares and Standard Errors

The sum of squares for error is equal to

$$\sum_{ij}\sum y_{ij}^2 - R(\alpha, a_i, b) .$$

The reduction due to fitting all constants $R(\alpha, a_i, b)$ is equal to $R(\mu, a_i, b)$, and is obtained from

$$\hat{\alpha}\, Y_. + \sum_i \hat{a}_i Y_i + \hat{b} \sum_{ij}\sum X_{ij} y_{ij}$$

or more easily from

$$\hat{\alpha}\, Y_. + \sum_{i=1}^{p-1} \hat{a}_i (Y_i - Y_p) + \hat{b} \sum_{ij}\sum X_{ij} y_{ij} .$$

If the X_{ij} have been expressed as deviations from $\bar{x}$, then $\hat{\mu}$ is obtained directly rather than $\hat{\alpha}$, and

$$R(\mu, a_i, b) = R(\alpha, a_i, b) = \hat{\mu}\, Y_. + \sum_i \hat{a}_i Y_i + \hat{b} \sum_{ij}\sum (X_{ij} - \bar{x})\, y_{ij}$$

$$= \hat{\mu}\, Y_. + \sum_{i=1}^{p-1} \hat{a}\, (Y_i - Y_p)$$

$$+ \hat{b} \sum_{ij}\sum (X_{ij} - \bar{x}) y_{ij}$$

In the one-way classification with covariance the error sum of squares can be obtained more directly, of course, from

$$\sum_{ij}\sum y_{ij}^2 - \sum_i \frac{Y_i^2}{n_i} - b^2\left(\sum_{ij}\sum X_{ij}^2 - \sum_i \frac{X_i^2}{n_i}\right) ,$$

since $R(\mu, a_i, b)$ is equal to the last two terms in this formula.

The sum of squares among the A classes, adjusted for variations in the mean of X, may be obtained from $B'Z^{-1}B$ as explained in the previous section.

In the one-way classification with covariance this sum of squares is obtained by direct covariance procedures. The more general methods are presented here so that extension to analyses where standard methods of analysis are unavailable can be made more easily.

The sum of squares for testing the significance of the regression coefficient (from zero) is obtained in the same manner, i.e., $B'Z^{-1}B$, but in this case the seemingly complicated matrix multiplication reduces to

$$\hat{b}^2 \left(\sum_{ij}\sum X_{ij}^2 - \sum_i \frac{X^2_{i}}{n_i}\right) .$$

Standard errors for the least-squares means for the A classes, $\hat{\mu} + \hat{a}_i$, may be obtained from the variance-covariance inverse matrix for α, a_i, and b as follows:

$$s_{\hat{\mu}+\hat{a}_i} = \sqrt{(C^{\alpha\alpha} + \bar{X}^2 C^{bb} + C^{a_i a_i} + 2\bar{X}\, C^{\alpha b} + 2C^{\alpha a_i} + 2\bar{X}\, C^{b a_i})\, \hat{\sigma}_e^2}$$

where $\hat{\sigma}_e^2$ is the mean square for error and the superscripts on the C's identify the inverse elements.

The inverse elements that would have been obtained for the μ-row (or-column) if the X's had been taken as deviations from $\bar{X}$ can be computed, if desired, from

$$C^{\mu\mu} = C^{\alpha\alpha} + 2\bar{X}\, C^{\alpha b} + \bar{X}^2\, C^{bb}$$

$$C^{\mu a_i} = C^{\alpha a_i} + \bar{X}\, C^{b a_i} .$$

If these are available, the standard errors of the $\hat{\mu} + \hat{a}_i$ can easily be computed from

$$s_{\hat{\mu}+\hat{a}_i} = \sqrt{(C^{\mu\mu} + C^{a_i a_i} + 2C^{\mu a_i})\, \hat{\sigma}_e^2}$$

Computation of standard errors of differences among the least-squares means need not involve the inverse elements in the μ or a row (or column) since

$$s_{(\hat{\mu}+a_i) - (\hat{\mu}+\hat{a}_j)} = s_{\hat{a}_i - \hat{a}_j} = \sqrt{(C^{a_i a_i} + C^{a_j a_j} - 2C^{a_i a_j})\, \hat{\sigma}_e^2}$$

These standard errors of differences among the $\hat{a}_i$ are required if mean seperation procedures are to be employed. The inverse elements for the $\hat{a}_i$ section can be transformed, as explained in the previous section, if orthogonal comparisons among the $\hat{a}_i$ are desired.

Numerical Example

The same set of data used in the previous section is repeated below but with the addition of a covariate as follows:

	Ration No.					
	1		2		3	
Pig No.	W	y	W	y	W	y
1	5	3	4	5	8	7
2	9	5	7	6	7	6
3	11	6	0	2	3	4
4	3	2	8	7	2	3
5			10	8	8	6
6			2	3	2	4
7			12	9		
8			5	8		
Totals	28	16	48	48	30	30
Means	7	4	6	6	5	5

1) Mathematical model:

$$y_{ij} = \alpha + r_i + b\, W_{ij} + e_{ij}$$

$$i = 1, 2, 3$$

$$j = 1, 2, \ldots, 8$$

where:

y_{ij} = the gain of the j^{th} barrow on the i^{th} ration,

α = the overall mean when $W_{ij} = 0$,

r_i = the effect of the i^{th} ration,

b = partial regression of gain on initial weight,

W_{ij} = the initial weight for the j^{th} pig on the i^{th} ration,

e_{ij} = random errors.

It should be pointed out that the assumption of random errors requires that the regression be linear and homogeneous from one ration to another.

2) Least-squares equations:

	$\hat{\alpha}$	$\hat{r}_1$	$\hat{r}_2$	$\hat{r}_3$	$\hat{b}$	RHM
α:	18	4	8	6	106	94
r_1:	4	4	0	0	28	16
r_2:	8	0	8	0	48	48
r_3:	6	0	0	6	30	30
b:	106	28	48	30	832	656

3) Imposing the restriction that $\sum_i \hat{r}_i = 0$:

The reduced least-squares equations are given below in tabular form:

	$\hat{\alpha}$	$\hat{r}_1$	$\hat{r}_2$	$\hat{b}$	RHM
α:	18	-2	2	106	94
r_1:	-2	10	6	-2	-14
r_2:	2	6	14	18	18
b:	106	-2	18	832	656

4) Inverse of the reduced least-squares equations and constant estimates:

The inverse of the reduced set of least-squares equations obtained with an ordinary method of inversion is given below:

Matrix Inverse

	$\hat{\alpha}$	$\hat{r}_1$	$\hat{r}_2$	$\hat{b}$
α:	.242003	.053451	-.018519	-.030303
r_1:	.053451	.148569	-.064815	-.005051
r_2:	-.018519	-.064815	.101852	.000000
b:	-.030303	-.00505[illegible]	.000000	.005051

. . . . . . (2)

The constant estimates may be obtained by multiplying each column (or row) of the inverse by the reduced RHM's. The constants obtained are:

$$\hat{\alpha} = 1.7879 \qquad \hat{r}_2 = 1.0000$$

$$\hat{r}_1 = -1.5354 \qquad \hat{b} = .5354$$

The overall mean $\hat{\mu}$ is then obtained from

$$\hat{\mu} = \hat{\alpha} + \hat{b}\,\bar{\bar{w}}$$

$$= 1.7879 + (.5354)(5.8889)$$

$$= 4.9408$$

The estimate of r_3 is

$$\hat{r}_3 = -(\hat{r}_1 + \hat{r}_2)$$

$$= -(-1.5354 + 1.000)$$

$$= .5354 \ .$$

Since,

$$\hat{\mu} + \hat{r}_i = \hat{a} + \hat{b}\,\hat{w} + \hat{r}_i \ ,$$

the diagonal inverse element for $\hat{\mu} + \hat{r}_i$ can be computed from

$$C^{\mu+r_i} = C^{aa} + \bar{w}^2 C^{bb} + C^{r_i r_i} + 2\bar{w}\, C^{ab} + 2C^{ar_i} + 2\bar{w}\, C^{br_i} \ .$$

Another method of computing this inverse diagonal element, which is necessary if the standard error of $\hat{\mu} + \hat{r}_i$ is desired, is possible from

$$C^{\mu\mu} + C^{r_i r_i} + 2C^{\mu r_i} \ .$$

However, the inverse elements involving μ are not directly available from the inverse matrix unless the W_{ij} have been expressed as deviations from the overall mean. These inverse elements for the μ-row and-column can be computed from the available inverse elements, as follows:

$$C^{\mu\mu} = C^{aa} + 2\bar{w}\, C^{ab} + \bar{w}^2\, C^{bb}$$

$$C^{\mu r_i} = C^{ar_i} + \bar{w}\, C^{br_i} \ .$$

In the present problem,

$$C^{\mu\mu} = .242003 + (2)(5.888889)\,(-.030303) + (5.888889)^2\,(.005051)$$

$$= .060265 \ ,$$

$$C^{\mu r1} = .053451 + (5.888889)(-.005051)$$

$$= .023706 \ ,$$

$$C^{\mu r2} = -.018519 + (5.888889)\,(.000000)$$

$$= -.018519 \ .$$

(a) Alternative procedure for computing the inverse matrix. As indicated above, a method of obtaining the inverse of the variance-covariance matrix more easily in some cases is available when constants for all degrees of freedom among the classes or subclasses are being fitted. This method was described above for the one-way classification when a regression coefficient is being fitted and will now be illustrated with the computational example.

The inverse element for the b, $\hat{b}$ position of the inverse matrix is

$$C^{bb} = \frac{1}{\sum\limits_{ij}\sum W_{ij}^2 - \sum\limits_i \frac{W_{i}^2}{n_i}}$$

$$= \frac{1}{198}$$

$$= .00505051$$

which checks with the value for this element obtained in the simultaneous inversion of the complete matrix. The inverse elements for the b-column and-row for the $\alpha + r_i$ are obtained as follows:

$$G^{1b} = G^{b1} = -\frac{W_1}{n_1\left[\sum_{ij}\sum W_{ij}^2 - \sum_i \frac{W_i}{n_i}\right]}$$

$$= -(.00505051)\left(\frac{28}{4}\right)$$

$$= -.03535357\ ,$$

$$G^{2b} = G^{b2} = -(.00505051)\left(\frac{48}{8}\right)$$

$$= -.03030306,$$

$$G^{3b} = G^{b3} = (.00505051)\left(\frac{30}{6}\right)$$

$$= -.02525255\ .$$

The adjusted diagonal inverse elements for $\hat{\alpha} + r_i$ are then computed as follows:

$$A^{11} = \frac{1}{n_1}(1 - W_1 G^{b1})$$

$$= \frac{1}{4}\left[1 - (28)(-.03535357)\right]$$

$$= .49747499\ ,$$

$$A^{22} = \frac{1}{8}\left[1 - (48)(-.03030306)\right]$$

$$= .30681836\ ,$$

$$A^{33} = \frac{1}{6}\left[1 - (30)(-.02525255)\right]$$

$$= .29292942.$$

The adjusted off-diagonal inverse elements for the $\alpha + r_i$ (which were zero) are then computed as follows:

$$A^{12} = A^{21} = -\frac{1}{n_i} W_1\ G^{b2}$$

$$= -\frac{1}{4}(28)(-.03030306)$$

$$= .21212142\ ,$$

$$A^{13} = A^{31} = -\frac{1}{4}(28)(-.02525255)$$

$$= .17676785 ,$$

$$A^{23} = A^{32} = -\frac{1}{8}(48)(-.02525255)$$

$$= .15151530.$$

Transformation of the $\alpha + r_i$ segment of the inverse matrix is now accomplished as follows:

$$K\acute{A}K' = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} .49747499 & .21212142 & .17676785 \\ .21212142 & .30681836 & .15151530 \\ .17676785 & .15151530 & .29292942 \end{bmatrix} K'$$

$$= \frac{1}{3}\begin{bmatrix} .88636426 & .67045508 & .62121257 \\ .60606071 & -.03409082 & -.09090902 \\ -.25000000 & .25000000 & -.16666667 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 2 \\ 1 & -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} .24200355 & .05345121 & -.01851852 \\ .05345121 & .14856903 & -.06481481 \\ -.01851852 & -.06481481 & .10185185 \end{bmatrix} .$$

These inverse elements check, within rounding errors, with those obtained directly (see matrix inverse (2)) from matrix inversion procedures. Actually, less rounding errors will accumulate in the computation of the inverse elements with this indirect method than when the inversion is completed on the least-squares equations by an elimination method.

The transformed inverse elements for the b-column (and row) are obtained from

$$KG = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} -.03535357 \\ -.03030306 \\ -.02525255 \end{bmatrix}$$

$$= \begin{bmatrix} -.03030306 \\ -.00505051 \\ .00000000 \end{bmatrix} ,$$

which also check with the values for these elements that were obtained directly (see matrix inverse (2)).

(5) The analysis of variance:

(a) Sum of squares for error.

$$\text{Error S. Sqs.} = \sum_{ij}\sum y_{ij}^2 - R(\alpha, r_i, b)$$

$$= 568 - (1.7879)(94) - (-1.5354)(-14)$$

$$-(1.0000)(18)-(.5354)(656)$$

$$= 568 - 558.7806 = 9.2194.$$

(b) Sum of squares for rations (R).

$$\text{Ration S.Sqs.} = B'Z^{-1}B$$

$$= \begin{bmatrix} -1.5354 & 1.0000 \end{bmatrix} \begin{bmatrix} .148569 & -.064815 \\ -.064815 & .101852 \end{bmatrix}^{-1} \begin{bmatrix} -1.5354 \\ 1.0000 \end{bmatrix}$$

$$= \begin{bmatrix} -1.5354 & 1.0000 \end{bmatrix} \begin{bmatrix} 9.317664 & 5.929431 \\ 5.929431 & 13.591447 \end{bmatrix} \begin{bmatrix} -1.5354 \\ 1.0000 \end{bmatrix}$$

$$= \begin{bmatrix} -8.376910 & 4.487399 \end{bmatrix} \begin{bmatrix} -1.5354 \\ 1.0000 \end{bmatrix}$$

$$= 17.3493$$

(c) Sum of squares due to regression.

$$\text{Regression S.Sqs.} = B'Z^{-1}B$$

$$= \frac{(.5354)^2}{.005051} = (.5354)^2 \, (198)$$

$$= 56.7573$$

The analysis of variance is as follows:

Source of Variation	d.f.	S.Sqs.	M.S.	F
Rations	2	17.3493	8.6746	13.17**
Regression	1	56.7573	56.7573	86.19**
Error	14	9.2194	.6585	

Differences due to rations are now highly significant; whereas in the first section, using the same data but ignoring the initial weight, ration differences were not significant. The error term has been reduced from 4.4000 to 0.6585.

(6) Standard errors and individual comparisons:

(a) Standard errors of the least squares means, $\hat{\mu}$ and $\hat{\mu} + \hat{r}_i$.

$$s_{\hat{\mu}} = \sqrt{(.060265)(.6585)}$$

$$= .199$$

$$s_{\hat{\mu}+\hat{r}_i} = \sqrt{[.060265 + .148569 + (2)(.023706)] \; (.6585)}$$

$$= .411$$

$$s_{\hat{\mu}+\hat{r}_2} = \sqrt{[.060265 + .101852 + (2)(-.018519)](.6585)}$$

$$= .287 ,$$

$$s_{\hat{\mu}+\hat{r}_3} = \sqrt{[.060265 + .120791 + (2)(-.005187)](.6585)}$$

$$= .335$$

(b) Individual comparisons among the $\hat{r}_i$.

(i) "t" test

$$\hat{r}_1-\hat{r}_2 = -2.5354 \qquad t = -\frac{2.5354}{.5003} = 5.068^{**},$$

$$\hat{r}_1-\hat{r}_3 = -2.0708 \qquad t = -\frac{2.0708}{.5364} = 3.861^{**},$$

$$\hat{r}_2-\hat{r}_3 = \quad .4646 \qquad t = \frac{.4646}{.4420} = 1.051.$$

(ii) Mean separation with Duncan's Multiple Range Test (.05).

Comparison	$\bar{y}_i-\bar{y}_j$	$\sqrt{\frac{2}{c^{ii}+c^{jj}-2c^{ij}}}$	Product Differences	$\hat{\sigma}_e\, z_{p_1n_2}$
$\hat{r}_2$ vs $\hat{r}_3$	.4646	2.596	1.21	2.46
$\hat{r}_2$ vs $\hat{r}_1$	2.5354	2.294	5.82*	2.58
$\hat{r}_3$ vs $\hat{r}_1$	2.0708	2.140	4.43*	2.46

(iii) Orthogonal contrasts.

Suppose that it is desired to test the significance of linear and quadratic effects among the $\hat{r}_i$. In this case the transformation matrix for the r_i segment of the inverse matrix (including the third row and column) is,

$$K = \frac{1}{4}\begin{bmatrix} 2 & 0 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

and

$$T = KZK' = \frac{1}{4}\begin{bmatrix} 2 & 0 & -2 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} \underline{.148569} & -.064815 & -.083754 \\ -.064815 & \underline{.101852} & -.037037 \\ -.083754 & -.037037 & \underline{.120791} \end{bmatrix} K'$$

$$= \frac{1}{4}\begin{bmatrix} .464646 & -.055556 & -.409090 \\ .194445 & -.305556 & .111111 \end{bmatrix} \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 0 & -2 \\ -2 & 1 \end{bmatrix}$$

$$= \frac{1}{16}\begin{bmatrix} 1.747472 & .166668 \\ .166668 & .916668 \end{bmatrix}$$

$$= \begin{bmatrix} .109217 & .010417 \\ .010417 & .057292 \end{bmatrix} .$$

$$\text{S.Sqs. Linear} = \frac{\left[\frac{1}{4}(2)(-1.5354) - \frac{1}{4}(2)(.5354)\right]^2}{.109217}$$

$$= \frac{(1.0354)^2}{.109217} = 9.816^{**} .$$

$$\text{S.Sqs. Quadratic} = \frac{\left[\frac{1}{4}(-1.5354) - \frac{1}{4}(2)(1.0000) + \frac{1}{4}(.5354)\right]^2}{.057292}$$

$$= \frac{(-.7500)^2}{.057292} = 9.818^{**} .$$

Again, it will be noted that the total of the sums of squares for the orthogonal contrasts is not equal to the sum of squares for rations.

Summary of the One-way Classification Analysis with Covariance

The least-squares procedures (method of fitting constants) for analyzing data with unequal numbers have been applied to the analysis of data classified in only one way but also considering a continuous covariate. Detailed generalized procedures have been presented for the computation of the variance-covariance inverse matrix, the constant estimates, sums of squares for the analysis of variance, standard errors of least-squares means and tests of significance for individual comparisons or for orthogonal contrasts among the class effects. These procedures are first presented in algebraic form for this design and then numerically using a simple example. The basic procedures described here are directly applicable to more complex designs which will be discussed in later sections.

TWO-WAY CLASSIFICATION WITHOUT INTERACTION

When disproportionate numbers exist among subclasses in the two-way classification the method of fitting constants by least-squares should usually be employed. For the special case of only two classes of A or two classes of B, short-cut methods for obtaining sums of squares for the analysis of variance have been developed and are given in several statistical books. The least-squares method of fitting constants for the general case of the two-way classification without interaction will be presented here.

Model

The model for the two-way classification when the interaction of A and B is assumed non-existent is

$$y_{ijk} = \mu + a_i + b_j + e_{ijk} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

$$i = 1, 2, \ldots, p$$

$$j = 1, 2, \ldots, q$$

where:

y_{ijk} = the k^{th} observation in the j^{th} B class and the i^{th} A class,

μ = overall mean when equal subclass numbers exist,

a_i = effect of the i^{th} A class,

b_j = effect of the j^{th} B class,

e_{ijk} = random errors, assumed to be $NID(0,\sigma_e^2)$.

This model reduces to that for the randomized block design only when k=1 for each combination of i and j.

The a_i and the b_j may either be regarded as fixed or random effects. If both classes of effects are fixed, the model is referred to as the "fixed" model, the linear hypothesis model, or Model I of Eisenhart. If both the a_i and the b_j are randomly drawn from some infinite population, such as cows, dams, sires, etc., the model is referred to as the "random" model or Model II of Eisenhart. If the effects for one of the classes are fixed and the other random, the model is referred to as the "mixed" model.

If a class of effects is to be regarded as fixed (such as the effects of a set of selected treatments), then the investigator is interested in least-squares means, standard errors, tests of significance, mean separation procedures and possibly orthogonal comparisons. On the other hand, if a class of effects is regarded as random, the investigator is primarily interested in estimating the variance component from that source. Hence, with random effects it is desirable to have a large number of degrees of freedom among the effects in order to estimate the variance (or possibly covariance) component more accurately. In both cases, however, it is desirable to obtain the least-squares sums of squares. With fixed effects, it is essential to obtain the inverse of the variance-covariance matrix in order to make individual comparisons. In many cases, it is also desirable to compute the inverse segment(s) of the variance-covariance matrix for the random effects. The general least-squares procedure for both kinds of effects will therefore be presented and followed by an absorption procedure which reduces the number of computations when the random effects are associated with a large number of

degrees of freedom.

Least-Squares Equations

The least-squares equations for the two-way classification are given in tabular form below:

	$\hat{\mu}$	$\hat{a}_i$	$\hat{b}_j$	RHM
μ:	$n_{..}$	$n_{i.}$	$n_{.j}$	$Y_{..}$
a_i:	$n_{i.}$	$_0\, n_{i.}\, ^0$	n_{ij}	$Y_{i.}$
b_j:	$n_{.j}$	n_{ij}	$_0\, n_{.j}\, ^0$	$Y_{.j}$

It should be noted that the sum of the coefficients for the $\hat{a}_i$ in the μ equation equals the sum of the coefficients for the $\hat{b}_j$ and the coefficient for $\hat{\mu}$. In addition, the sum of the coefficients for the $\hat{b}_j$ in an a_i equation equals the coefficient for the a_i and the total of the RHM's for the a_i equations and the b_j equations equals the grand total of the y_{ij}, $Y_{..}$. In order to solve these equations or to invert the coefficient matrix it is necessary to impose restrictions on the $\hat{a}_i$ and one on the $\hat{b}_j$.

Imposing Restrictions

A common restriction on these equations is to set $a_p = b_q = 0$ and delete the equations and columns for a_p and b_q. The inverse of the resulting reduced coefficient-matrix must be transformed if the standard errors of the $\hat{\mu} + \hat{a}_i$ or the $\hat{\mu} + \hat{b}_j$ are desired, or if the coefficients of variance components in the expectation of mean squares are to be obtained by means of a short-cut procedure which will be described. Hence, it is generally preferred to impose the restrictions that $\sum_i \hat{a}_i = \sum_j \hat{b}_j = 0$ and to make the necessary subtractions in the least-squares equations before inversion of the matrix. It is much easier to complete these necessary subtractions before inversion than to transform the inverse matrix. In addition, the constants obtained from the direct solution of the reduced matrix will be in the form desired, in most cases, i.e., $\hat{\mu}$ is estimated directly rather than as $\hat{\mu} + \hat{a}_p + \hat{b}_q$, the $\hat{a}_i$ are estimated directly rather than as $\hat{a}_i - \hat{a}_p$, and the $\hat{b}_j$ are estimated directly rather than as $\hat{b}_j - \hat{b}_q$.

When the restrictions are imposed that $\sum_i \hat{a}_i = \sum_j \hat{b}_j = 0$ the coefficients of one equation in the a_i, say a_p, and one equation in the b_j, say b_q, must be subtracted from other coefficients by columns and rows. The subtraction of the $\hat{a}_p$ coefficients is done only within the $\hat{a}_i$ columns of coefficients and

the subtraction of the $\hat{b}_q$ coefficients is done only in the $\hat{b}_j$ columns. After completion of this subtraction by columns, the coefficients and RHM in the resulting a_p equation are subtracted from the corresponding coefficients and the RHM's in the other a_i equations. A similar procedure is followed for the b_q equation coefficients. After these subtractions are completed there will remain $1 + p - 1 + q - 1$ or $p + q - 1$ symmetrical equations. The number of remaining equations is also the number of degrees of freedom among the $\hat{a}_i$, among the $\hat{b}_j$ and one additional for $\hat{\mu}$.

Inversion of the Reduced Matrix and Solution of the Equations

A least-squares matrix for the two-way classification can usually be inverted without difficulty using ordinary procedures and in single precision arithmetic (i.e., using 7 or 8 significant digit arithmetic) unless confounding of the effects is complete in one or more cases. With experience it becomes easy to recognize complete confounding of an a_i and a b_j effect (if such should exist) from the n_{ij} section of the original least-squares equations.

The reduced set of least-squares equations can usually be solved directly to obtain estimates of the constants, μ, a_i, and b_j, at the same time the inversion is being made. The $\hat{a}_p$ and $\hat{b}_q$ constants are obtained from

$$\hat{a}_p = -\sum_i \hat{a}_i$$

$$\hat{b}_q = -\sum_j \hat{b}_j$$

and the inverse elements for the a_p and b_q columns and rows may be obtained in a manner similar to that explained in the first section.

Computing Sums of Squares for the Analysis of Variance

The total reduction in sum of squares is

$$R(\mu, a_i, b_j) = \hat{\mu}\, Y_{..} + \sum_{i=1}^{p-1} \hat{a}_i \,(Y_{i.} - Y_{p.}) + \sum_{j=1}^{q-1} \hat{b}_j \,(Y_{.j} - Y_{.q})$$

and the error sum of squares is equal to

$$\sum_i\sum_j\sum_k y^2_{ijk} - R(\mu, a_i, b_j)$$

when interaction effects are non-existent.

The least-squares or adjusted sum of squares for testing the significance of differences among the A classes can most easily be computed in the two-way classification from $R(\mu, a_i, b_j) - R(\mu, b_j)$, where

$$R(\mu, b_j) = \sum_j \frac{Y_{.j}^2}{n_{.j}},$$

the "between" uncorrected sum of squares for B. Likewise, the least-squares sum of squares for testing the significance of the differences among the B classes is obtained from

$$R(\mu, a_i, b_j) - R(\mu, a_i), \text{ where}$$

$$R(\mu, a_i) = \sum_i \frac{Y_{i.}^2}{n_{i.}},$$

the "between" uncorrected sum of squares for A. These sums of squares are also computed when variance components are to be estimated. In this case, coefficients for variance components in the expectation of these sums of squares must be computed. Methods for making these computations are considered in a later sub-section.

In the two-way classification analysis, the least-squares sum of squares for the interaction of A and B can be computed indirectly even though the interaction constants are not fitted. This is true because the sum of squares for interaction is equal to $R[\mu, a_i, b_j, (ab)_{ij}] - R(\mu, a_i, b_j)$ and

$$R[\mu, a_i, b_j, (ab)_{ij}] = \sum_{ij}\sum \frac{Y_{ij}^2}{n_{ij}},$$

the "between" uncorrected sum of squares for the AB subclasses. The same is true in least-squares analyses with more complex models. That is, when constants for all effects among the subclasses are fitted, except the highest order interaction, the sum of squares for this interaction can be obtained indirectly from the difference between the subclass sum of squares and the total reduction due to fitting constants. A similar indirect procedure can be used to compute the interaction when partial regressions for continuous variates are also fitted at the same time, provided constants for all effects among the subclasses have been fitted except the highest order interaction.

The error sum of squares for testing the significance of interaction effects is

$$\sum_{ijk}\sum\sum y_{ijk}^2 - \sum_{ij}\sum \frac{Y_{ij}^2}{n_{ij}}$$

the "within" subclass sum of squares.

When the interaction effects are found to be significant, by means of the test of significance in the analysis of variance, the main effects and the corresponding sums of squares are biased. This is true because of the partial confounding of the main effects and the interaction effects due to unequal subclass numbers. If interaction effects exist the main effects must

therefore be estimated from averaging over the subclass means rather than from the class means. When interaction effects exist in the two-way classification and all subclasses are filled, the weighted squares of means procedure, described in many textbooks on statistics, may be used to compute the sums of squares for the main effects adjusted for the interaction. The weighted squares of means analysis is equivalent to the complete least-squares analysis when constants have been fitted for all main effects and interactions. This will be shown to be the case in the next section where consideration is given to the fitting of interaction constants.

When all subclasses of A and B are not filled, the degrees of freedom for the interaction sum of squares must be obtained by difference rather than from $(p-1)(q-1)$. The number of degrees of freedom for interaction in any case is equal to the number of AB subclasses minus the number of A classes minus the number of B classes plus one.

Standard Errors, Orthogonal Comparisons and Mean Separation Procedures

Detail procedures for computing standard errors of constant estimates and linear functions of constants estimates are the same as those described in the previous sections for the one-way classification. Likewise, methods for testing the significance of pairwise differences among least-squares means and orthogonal contrasts as used in the one-way classification are also applicable to the two-way classification. In either case the inverse of the variance-covariance matrix is required.

Estimation of Variance Components

When the $\hat{a}_i$ and the $\hat{b}_j$ are random samples from populations of these effects, the estimates of the variance components associated with these effects are of primary interest. These variance component estimates are obtained by equating the least-squares sums of squares or mean squares to their corresponding expectations and solving the resulting equations. In the two-way classification without interaction, the analysis of variance is as follows:

Source of Variation	d.f.	S.Sqs.	E(MS)
A	$p-1$	$R(\mu,a_i,b_j) - R(\mu,b_j)$	$\sigma_e^2 + k_2\sigma_a^2$
B	$q-1$	$R(\mu,a_i,b_j) - R(\mu,a_i)$	$\sigma_e^2 + k_1\sigma_b^2$
Error	$n_{..}-p-q+1$	$\sum_{ijk}\sum\sum y_{ijk}^2 - R(\mu,a_i,b_j)$	σ_e^2

The coefficients of the variance components in the expectation of the mean squares, E(MS), may be computed with either of two methods. One method (the first to be described) may be regarded as the direct method and the other as an indirect method. The direct method of computing the coefficients of variance components, the k's, parallels the direct method of computing the sums of squares and the indirect method of computing these coefficients parallels

the indirect method of computing the sums of squares.

It will be recalled that any least-squares sum of squares may be computed directly from the matrix multiplication $B'Z^{-1}B$, where B' is a row vector of a set of constants, Z^{-1} is the inverse of the square symmetrical segment of the variance-covariance inverse corresponding to this set of constants and B is a column vector of the constants. This is true for the two-way classification as well as for any other least-squares analysis. However, in this case the sums of squares for A and B can most easily be computed indirectly, i.e., from differences in reductions in sums of squares as indicated above.

Direct Method of Computing the k's

If the restrictions are imposed that $\sum_i \hat{a}_i = \sum_j \hat{b}_j = 0$ in the original equations and the least-squares sums of squares are computed from $B'Z^{-1}B$, the variance component coefficients k_1 and k_2 are easily computed as follows:

$$k_1 = \frac{1}{q}\left(\sum_i Z_B^{ii} - \frac{1}{q-1} \sum\sum_{ij,\ i \neq j} Z_B^{ij}\right)$$

$$k_2 = \frac{1}{p}\left(\sum_i Z_A^{ii} - \frac{1}{p-1} \sum\sum_{ij,\ i \neq j} Z_A^{ij}\right)$$

The superscripts on Z identify the elements in the matrix inverse to the square symmetrical segment from the variance-covariance inverse matrix and the subscripts A or B identify the segments for the A effects and the B effects.

Extension of the direct method of computing the coefficient for the major variance component in each line in the E(MS) for more complex designs is straight-forward. In general, the formula for computing these coefficients is

$$k = \frac{1}{m}\left(\sum_i Z^{ii} - \frac{1}{\text{d.f.}} \sum\sum_{ij,\ i \neq j} Z^{ij}\right),$$

where d.f. is the number of degrees of freedom for that line in the analysis of variance and m is the number of classes or subclasses. This equation, however, is not generally applicable to interactions.

Indirect Method of Computing the k's [4]

The indirect method is accomplished by first computing the coefficient for the variance component in the reduction in sum of squares due to fitting all effects except the set being considered. This coefficient is then subtracted from n... and the result is divided by the degrees of freedom for that source of variation to obtain the coefficient for the variance component. The problem consists of computing the coefficient for the variance component in the appropriate reduction in sum of squares. This is accomplished by

4/ Henderson, C. R. Estimation of variance and covariance components. Biometrics 9: 226-252. 1953.

computing the sum of crossproducts between corresponding elements of two square matrices.

The inverse of the variance-covariance matrix when all effects except the set for which the variance component coefficient is to be computed, are included in the model is one of the matrices required for use of this indirect method. For example, in the two-way classification analysis, if the coefficient for σ_a^2 is desired, then the inverse of the variance-covariance matrix for the the model $y_{jk} = \mu + b_j + e_{jk}$ is required. The "associated sums" matrix is the second square matrix required for use of this indirect method and is computed from the matrix multiplication N N', where N is the segment of the original complete set of least-squares equations which contains the coefficients associating the effects under consideration with all others in the model. For example, in the two-way classification, N is the n_{ij} section of the least-squares equations to the right of the main diagonal and N' is the n_{ij} section to the left of the main diagonal. If the elements in the inverse matrix for the least-squares equations for the reduced model are R^{ij} and the elements in the product matrix, N N', are N_{ij}, the coefficient for a variance component in the E(MS) of the analysis of variance is given by

$$\frac{n_{..} - \sum\limits_{ij}\sum R^{ij} N_{ij}}{\text{degrees of freedom}} .$$

The quantity

$$\sum_{ij}\sum R^{ij} \; N_{ij}$$

may also be computed from the sum of the diagonals in the matrix resulting from the multiplication

$$N'R^{-1}N \quad ,$$

where R^{-1} is the inverse of the variance-covariance matrix for the reduced model.

The indirect method of computing the coefficients for variance components is more easily completed than the direct method for the two-way classification analysis, as was the case in computing the sums of squares for tests of significance. The indirect method of computing the coefficients for σ_a^2 and σ_b^2 reduces to

$$k_1 = \frac{1}{q-1} \left(n_{..} - \sum_i \frac{\sum\limits_j n_{ij}^2}{n_{i.}}\right)$$

$$k_2 = \frac{1}{p-1} \left(n_{..} - \sum_j \frac{\sum\limits_i n_{ij}^2}{n_{.j}}\right)$$

for the two-way classification without interaction.

Absorption of the $\mu+a_i$ Equations

Although the inverse of the variance-covariance coefficient matrix for the complete model is required, it can often be obtained more easily by a partitioning procedure than from direct inversion of the reduced set of least-squares equations. The absorption of the $\mu+a_i$ equations into the b_j equations is only one step in this partitioning procedure.

It was shown in the one-way classification that the equations for $\mu+a_i$ are identical with the equations for the a_i. When μ is combined with the a_i it is unnecessary to impose a restriction on the a_i since there are p degrees of freedom associated with the $\mu+a_i$ but only p-1 degrees of freedom associated with the a_i alone. This being true, and with

$$\mu + a_i = \frac{1}{n_{i.}} (Y_{i.} - \sum_j n_{ij} b_j)$$

it is fairly easy to absorb the equations for $\mu + a_i$ into the equations for the b_j.

If the new coefficients for the b_j, after absorption of the $\mu + a_i$, are $C(b_j b_j)$ and $C(b_j b_{j'})$ and the new right hand members are $S(b_j)$, the reduced equations are as follows:

	$\hat{b}_1$	$\hat{b}_2$	$\hat{b}_3$		$\hat{b}_q$	RHM
b_1:	$C(b_1b_1)$	$C(b_1b_2)$	$C(b_1b_3)$	...	$C(b_1b_q)$	$S(b_1)$
b_2:	$C(b_2b_1)$	$C(b_2b_2)$	$C(b_2b_3)$	...	$C(b_2b_q)$	$S(b_2)$
b_3:	$C(b_3b_1)$	$C(b_3b_2)$	$C(b_3b_3)$	...	$C(b_3b_q)$	$S(b_3)$
.	.	.	.	...	.	.
.	.	.	.	...	.	.
.	.	.	.	...	.	.
b_q:	$C(b_qb_1)$	$C(b_qb_2)$	$C(b_qb_3)$	...	$C(b_qb_q)$	$S(b_q)$

The new coefficients and RHM's are computed as follows:

$$C(b_j b_j) = n_{.j} - \sum_i \frac{n_{ij}^2}{n_{i.}}$$

$$\underset{j \neq j'}{C(b_j b_{j'})} = C(b_{j'} b_j) = -\sum_i \frac{n_{ij} n_{ij'}}{n_{i.}}, \text{ and}$$

$$S(b_j) = Y_{.j} - \sum_i \frac{n_{ij} Y_{i.}}{n_{i.}}$$

Unless the number of $\mu + a_i$ equations is large and/or the number of b_j equations is large this absorption can be completed on an electric desk calculator. Programs for completing this absorption are now available for high speed electronic computers which permit a wide variation in p and q.

No restriction needs to be imposed on the least-squares equations until after absorption has been completed. After absorption it will be noted that the sum of the new coefficients for the b_j sum to zero by rows and columns and that the sum of the RHM's, by column, sum to zero. This provides a check on the absorption process, but does show that dependencies exist in the reduced set of equations. The dependencies are removed by imposing the restriction that $\sum_j \hat{b}_j = 0$, whereby the coefficients for $\hat{b}_q$ are subtracted from the coefficients of the other $\hat{b}_j$ and the resulting coefficients in the $\hat{b}_q$ equation are then subtracted from the resulting coefficients in the other b_j equations. The $S(b_q)$ is also subtracted from the other $S(b_j)$ in each of the RHM columns.

The inverse of the reduced set of equations for the b_j, after absorption of the $\mu + a_i$ and after imposing the appropriate restriction, is exactly the same as the inverse elements for the $b_j, \hat{b}_j$ segment in the inverse matrix of the complete variance-covariance matrix. The constants obtained for b_j are also identical to the b_j constants that would be obtained by solving the original set of equations. In effect, the constants for the b_j are fitted on a within A class basis in either case.

The reduction in sum of squares obtained by multiplying the $\hat{b}_j$ by the $S(b_j) - S(b_q)$ is equal to the least-squares sum of squares for B, i.e.,

$$\begin{aligned} \text{S.Sqs. B} &= \sum_{j=1}^{q-1} \hat{b}_j \left[S(b_j) - S(b_q)\right] \\ &= \sum_j \hat{b}_j \;\; S(b_j) \\ &= R(\mu, a_i, b_j) - R(\mu, a_i), \end{aligned}$$

Estimates of the $\mu + a_i$, the least-squares means for the A classes, are obtained by simply solving the equations for $\mu + a_i$ after computing the $\hat{b}_j$ constants. For example,

$$\hat{\mu} + \hat{a}_1 = \frac{1}{n_{1.}} (Y_{1.} - \sum_j n_{1j} \hat{b}_j)$$

and

$$\hat{\mu} + \hat{a}_i = \frac{1}{n_{i.}} (Y_{i.} - \sum_j n_{ij} \hat{b}_j)$$

The remaining inverse elements for the complete variance-covariance inverse may be obtained if desired from a relatively simple but general matrix multiplication procedure. If C is the inverse matrix of the set of equations remaining after absorption and after appropriate restrictions have been imposed, D^{-1} is the inverse of the diagonal matrix for the classes $(\mu + a_i)$ or subclasses absorbed and N_M is the modified matrix of off-diagonal coefficients in the original least-squares equations which associate the effects being absorbed with the remaining effects, then the inverse elements for the n_{ij}-section, (G), are obtained from $-D^{-1}N_MC$. In the two way classification without interaction, this is

$$G = -\begin{bmatrix} \frac{1}{n_{1.}} & 0 & \cdots & 0 \\ 0 & \frac{1}{n_{2.}} & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \frac{1}{n_{p.}} \end{bmatrix} \begin{bmatrix} n'_{11} & n'_{12} & \cdots & n'_{1(q-1)} \\ n'_{21} & n'_{22} & \cdots & n'_{2(q-1)} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ n'_{p1} & n'_{p2} & \cdots & n'_{p(q-1)} \end{bmatrix} \begin{bmatrix} C^{11} & C^{12} & \cdots & C^{1(q-1)} \\ C^{21} & C^{22} & \cdots & C^{2(q-1)} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ C^{(q-1)1} & C^{(q-1)2} & \cdots & C^{(q-1)(q-1)} \end{bmatrix},$$

where the n'_{ij} are obtained from the subtraction of the last column of the n_{ij} from each of the other n_{ij} columns, i.e.,

$$n'_{ij} = n_{ij} - n_{iq} .$$

If $\hat{b}_q$ was set equal to zero to obtain C, then the last column of the n_{ij} can be deleted rather than subtracted out as indicated above. However, in this case the inverse elements obtained do not apply directly to the constants as shown in the model (3). Also, the constants which are thus obtained directly are linear functions of μ, a_i, and b_j rather than the constants themselves.

The inverse elements for the $\mu + a_i$ square segment (A) are then obtained from

$$A = D^{-1} (I - N_MG')$$

where I is the identity or unit matrix and G' is the transpose of G.

Since D is a diagonal matrix in the absorption of a set of equations, the matrix multiplication procedures required to complete the inverse can be done rapidly - - even on a desk calculator in many cases. In addition, the use of this method to obtain the complete variance-covariance inverse often results in less accumulation of rounding errors. It is also possible with this procedure to selectively compute the inverse for the $\mu + a_i$ when only certain of these means are to be compared. If desired, the diagonal elements for the $\mu + a_i$ can be computed without computing the off-diagonal delments. This can

often be accomplished without much effort even though there are several hundred $\mu + a_i$ being estimated.

When the a_i effects are random it is often desirable to obtain the maximum likelihood estimate for the A classes. Henderson (1949) 5/ has shown that the maximum likelihood estimates can be obtained by regressing the least-squares means. The amount by which the least-squares mean must be regressed is a function of the corresponding diagonal inverse element, i.e.,

$$\hat{\mu} + \hat{\hat{a}} \doteq \hat{\mu} + \frac{\hat{\sigma}_a^2}{\hat{\sigma}_a^2 + A^{ii}\hat{\sigma}_e^2} (\hat{\mu} + \hat{a}_i - \hat{\mu})$$

$$\doteq \hat{\mu} + \frac{\hat{\sigma}_a^2}{\hat{\sigma}_a^2 + A^{ii}\hat{\sigma}_e^2} (\hat{a}_i) ,$$

where $\hat{\mu} + \hat{\hat{a}}_i$ is the regressed least-squares mean, i.e., the maximum likelihood estimate, and the A^{ii} are the diagonal inverse elements for the $\hat{\mu} + \hat{a}_i$. Henderson shows that the maximum likelihood estimate reduces to

$$\hat{\mu} + \hat{\hat{a}}_i \doteq \hat{\mu} + \frac{nr}{1+(n-1)r} (\hat{a}_i)$$

for the one-way classification, where $\underline{n}$ is the number of observations in the i^{th} A class and $\underline{r}$ is the simple **intraclass** correlation or repeatability. The reason why these formulas are only approximations of the maximum likelihood estimates, rather than the exact formulas, is because μ and the σ^2's must be known without error to obtain the maximum likelihood estimate. Since this is impossible in practice, the estimates of μ and the σ^2's are used to obtain the "best" unbiased estimate.

Numerical Example

The same data used for the numerical example in the two previous sections will be used here by rearranging them as follows:

5/ Henderson, C. R. Estimation of general, specific and maternal combining abilities in crosses among inbred lines of swine. Unpublished Ph.D. thesis. Iowa State College Library. 199 p.p.

Ration No.	Pig No.	Sire No. 1	2	3
1	1	5	2	3
	2	6	3	-
	3	-	5	-
	4	-	6	-
	5	-	7	-
Sub-totals		11	23	3
2	1	2	8	4
	2	3	8	4
	3	-	9	6
	4	-	-	6
	5	-	-	7
Sub-totals		5	25	27
Totals		16	48	30
Means		4	6	5

1) Mathematical model:

$$y_{ijk} = \mu + s_i + r_j + e_{ijk}$$

$$i = 1, 2, 3$$

$$j = 1, 2$$

$$k = 1, 2, \ldots, n_{ij}$$

where:

y_{ijk} = the gain of the k^{th} barrow on the j^{th} ration by the i^{th} sire,

μ = the overall mean with equal subclass frequencies,

s_i = effect of the i^{th} sire,

r_j = effect of the j^{th} ration,

e_{ijk} = random errors.

2) Least-squares equations:

	$\hat{\mu}$	$\hat{s}_1$	$\hat{s}_2$	$\hat{s}_3$	$\hat{r}_1$	$\hat{r}_2$	RHM
μ:	18	4	8	6	8	10	94
s_1:	4	4	0	0	2	2	16
s_2:	8	0	8	0	5	3	48
s_3:	6	0	0	6	1	5	30
r_1:	8	2	5	1	8	0	37
r_2:	10	2	3	5	0	10	57

It will be noted that a number of dependencies exist in these equations. For example, in the μ equation the sum of the coefficients for $\hat{s}_i$ equals the sum of the coefficients for the $\hat{r}_j$, the overall number of observations. Likewise, the sum of the right-hand members for the s_i and the r_j equations each equal the right-hand member for the μ equation. Hence, the determinant of the coefficient matrix is zero and no unique solution or inversion is possible without imposing some restrictions on the equations.

3) Imposing the restrictions that $\sum_i \hat{s}_i = \sum_j \hat{r}_j = 0$:

	$\hat{\mu}$	$\hat{s}_1$	$\hat{s}_2$	$\hat{r}_1$	RHM
μ:	18	-2	2	-2	94
s_1:	-2	10	6	4	-14
s_2:	2	6	14	6	18
r_1:	-2	4	6	18	-20

4) Inverse of the reduced least-squares coefficient matrix and constant estimates:

The matrix inverse to the reduced coefficient matrix is as follows:

	$\hat{\mu}$	$\hat{s}_1$	$\hat{s}_2$	$\hat{r}_1$
μ:	.061486	.021848	-.022160	.009363
s_1:	.021848	.144819	-.061174	-.009363
s_2:	-.022160	-.061174	.112047	-.026217
r_1:	.009363	-.009363	-.026217	.067416

. . . . (4)

The constant estimates obtained by multiplying the inverse and RHM's of the reduced matrix together are

$$\hat{\mu} = 4.8876 \qquad \hat{s}_2 = 1.3146$$

$$\hat{s}_1 = -.8876 \qquad \hat{r}_1 = -.8090$$

Estimates of s_3 and r_2 are then computed as follows:

$$\hat{s}_3 = -(-.8876 + 1.3146)$$
$$= -.4270$$
$$\hat{r}_2 = -(-.8090) = .8090.$$

5) Sums of squares for the analysis of variance:

$$R(\mu, s_i, r_j) = (4.8876)(94) + (-.8876)(-14) + (1.3146)(18) + (-.8090)(-20)$$
$$= 511.7036.$$

The total uncorrected sum of squares for y_{ijk} is $\sum_{ijk}\sum\sum y^2_{ijk} = 568$.

Error S.Sqs. = 568 - 511.7036 = 56.2964 with no interaction.

$$\text{S.Sqs. Sires(S)} = R(\mu, s_i, r_j) - \sum_j \frac{Y^2_{.j}}{n_{.j}}$$

$$= 511.7036 - 496.0250 = 15.6786$$

$$\text{S.Sqs. Rations(R)} = R(\mu, s_i, r_j) - \sum_i \frac{Y^2_{i.}}{n_{i.}}$$

$$= 511.7036 - 502.0000 = 9.7036$$

$$\text{S.Sqs. Interaction (SR)} = \sum_{ij}\sum \frac{Y^2_{ij}}{n_{ij}} - R(\mu, s_i, r_j)$$

$$= 541.9333 - 511.7036 = 30.2297$$

$$\text{S.Sqs. Within SR Subclasses} = 568 - 541.9333$$

$$= 26.0667$$

The analysis of variance is as follows:

Source of Variation	d.f.	S.Sqs.1/	M.S.	F
Sires(S)	2	15.6786	7.8393	
Rations(R)	1	9.7036	9.7036	
SR	2	30.2297	15.1148	6.96**
Within Subclasses	12	26.0667	2.1722	

1/ The sums of squares for sires and rations are unadjusted for the interaction.

Since the test of significance for interaction is significant at the .01 level of probability, the least-squares analysis should be completed with constants being fitted for the interaction effects as well as the main effects. When all subclasses are filled, the appropriate least-squares analysis can best be completed for the two-way classification by the method of weighted squares of means. A discussion and the presentation of methods of analysis when the interaction must be considered is deferred until the next section. The analysis of these data will proceed as though no interaction effects existed.

With no interaction effects considered the analysis of variance is as follows:

Source of Variation	d.f.	S.Sqs.	M.S.	F
Sires(S)	2	15.6786	7.8393	1.95 n.s.
Rations(R)	1	9.7036	9.7036	2.41 n.s.
Error	14	56.2964	4.0212	

6) Standard errors and individual comparisons:

(a) Standard errors of least squares means.

$$s_{\hat{\mu}} = \sqrt{(.061486)(4.0212)}$$
$$= .50$$

$$s_{\hat{\mu}+\hat{s}_1} = \sqrt{[.061486 + .144819 + (2)(.021848)](4.0212)}$$
$$= 1.00.$$

$$s_{\hat{\mu}+\hat{s}_2} = \sqrt{[.061486 + .112047 + (2)(-.022160)](4.0212)}$$
$$= .72.$$

$$s_{\hat{\mu}+\hat{s}_3} = \sqrt{[.061486 + .134518 + (2)(.000312)](4.0212)}$$
$$= .89.$$

$$s_{\hat{\mu}+\hat{r}_1} = \sqrt{[.061486 + .067416 + (2)(.009363)](4.0212)}$$
$$= .77$$

$$s_{\hat{\mu}+\hat{r}_2} = \sqrt{[.061486 + .067416 + (2)(-.009363)](4.0212)}$$
$$= .67$$

(b) Individual comparisons among the $\hat{s}_i$.

(i) Mean separation with Duncan's Multiple Range Test (.05 level).

Comparison	$\bar{y}_i - \bar{y}_j$	$\sqrt{\dfrac{2}{c^{ii}+c^{jj}-2c^{ij}}}$	Product Differences	$\hat{\sigma}_e\ z_{p,n_2}$
$\hat{s}_2$ vs. $\hat{s}_3$	1.7416	2.396	4.17	6.08
$\hat{s}_2$ vs. $\hat{s}_1$	2.2292	2.297	5.12	6.38
$\hat{s}_3$ vs. $\hat{s}_1$	.4606	2.116	.97	6.08

These procedures are carried out on these data only for the purpose of showing the computations involved. In practice, if the F test in the analysis of variance is non-significant, it is doubtful that one would want to use a mean separation procedure.

(ii) Orthogonal comparisons among the $\hat{s}_i$.

The computational procedures involved in computing the least-squares sums of squares for orthogonal contrasts among a set of constants were illustrated in the two previous sections with the numerical example. Since this procedure is always the same regardless of the other constants being fitted in the least-squares analysis, it will not be shown here.

7) Estimation of variance components.

The number of degrees of freedom for sires and rations are inadequate, of course, to give accurate estimates of variance components for sires or for rations. However, in order to illustrate the computational procedures, which are directly applicable to analyses where the number of degrees of freedom are adequate, the effects of sires and rations will be regarded as random and estimates of σ_s^2 and σ_r^2 will be obtained.

The expectation of the mean square for rations is $\sigma_e^2 + k_1 \sigma_r^2$, and the expectation of the mean square for sires is $\sigma_e^2 + k_2\sigma_s^2$ when the interaction does not exist. Since the mean square for error contains only σ_e^2 the problem is to compute k_1 and k_2. As pointed out above, these k values may be computed by either the direct method or the indirect method. Although the indirect method is the easier to use in the two-way classification both methods will be presented here. Extension of the direct method to more complex analyses where it is more useful than the indirect method is straight-forward.

(a) Direct method of computing the k's.

The inverse matrices to the Z segments of the complete matrix inverse (4) are as follows:

$$Z_S^{-1} = \begin{bmatrix} .144819 & -.061174 \\ -.061174 & .112047 \end{bmatrix}^{-1} = \begin{bmatrix} 8.975050 & 4.900084 \\ 4.900084 & 11.600112 \end{bmatrix}$$

$$Z_R^{-1} = \begin{bmatrix} .067416 \end{bmatrix}^{-1} = \begin{bmatrix} 14.833274 \end{bmatrix}$$

The coefficients, k_1 and k_2, are now computed as follows:

$$k_1 = \frac{1}{2}\,(14.833274) = 7.417$$

$$k_2 = \frac{1}{3}\left[8.875050 + 11.600112 - \frac{1}{2}\,(4.900084 + 4.900084)\right]$$

$$= \frac{1}{3}\,(15.675078) = 5.225$$

The sums of squares for sires and rations may now be verified from $B'Z^{-1}B$.

$$\text{S.Sqs. Sires} = (-.8876 \quad 1.3146)Z_S^{-1}B$$

$$= (-1.5246 \quad 10.9002)\ B$$

$$= 15.683$$

$$\text{S.Sqs. Rations} = (-.8090)Z_R^{-1}\ B$$

$$= (-12.0001)\ B$$

$$= 9.708$$

These check, within rounding errors, with the sums of squares obtained with the indirect procedure.

(b) Indirect method of computing the k's.

Step 1. Computation of the matrix inverses to the reduced matrices.

The inverse of the coefficient matrix for $\hat{\mu} + \hat{s}_i$ is

$$\begin{array}{r} \\ \mu+s_1: \\ \mu+s_2: \\ \mu+s_3: \end{array} \begin{array}{c} \begin{array}{ccc} \hat{\mu}+\hat{s}_1 & \hat{\mu}+\hat{s}_2 & \hat{\mu}+\hat{s}_3 \end{array} \\ \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{6} \end{bmatrix} \end{array} = R^{ij}.$$

The inverse of the coefficient matrix for $\hat{\mu}+\hat{r}_j$ is

$$\begin{array}{r} \\ \mu+r_1: \\ \mu+r_2: \end{array} \begin{array}{c} \begin{array}{cc} \hat{\mu}+\hat{r}_1 & \hat{\mu}+\hat{r}_2 \end{array} \\ \begin{bmatrix} \frac{1}{8} & 0 \\ 0 & \frac{1}{10} \end{bmatrix} \end{array} = R^{ij}, \text{ or } \begin{bmatrix} 18 & -2 \\ -2 & 18 \end{bmatrix}^{-1} = \frac{1}{160} \begin{bmatrix} 9 & 1 \\ 1 & 9 \end{bmatrix}$$

Step 2. Computation of the "associated sums" matrices, NN'.

Rations associated with sires:

$$N_{ij} = N\,N' = \begin{bmatrix} 2 & 2 \\ 5 & 3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 2 & 5 & 1 \\ 2 & 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} \underline{8} & 16 & 12 \\ 16 & \underline{34} & 20 \\ 12 & 20 & \underline{26} \end{bmatrix}.$$

Sires associated with rations:

$$N_{ij} = N\,N' = \begin{bmatrix} 2 & 5 & 1 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 5 & 3 \\ 1 & 5 \end{bmatrix} \text{ or } \begin{bmatrix} 4 & 8 & 6 \\ 0 & 2 & -4 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 8 & 2 \\ 6 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} \underline{30} & 24 \\ 24 & \underline{38} \end{bmatrix}. \qquad = \begin{bmatrix} 116 & -8 \\ -8 & 20 \end{bmatrix}$$

Step 3. Computing the sum of products of R^{ij} and N_{ij}.

Sires:

$$\sum_{i}\sum_{j} R^{ij} N_{ij} = 10.583.$$

Rations:

$$\sum_{i}\sum_{j} R^{ij} N_{ij} = 7.550.$$

Step 4. Computing the k values.

$$k_1 = \frac{18-10.583}{1} = 7.417.$$

$$k_2 = \frac{18-7.550}{2} = 5.225$$

These k values check exactly with those computed by the direct method. With the two-way classification this indirect method reduces to the following:

$$k_1 = 18 - \frac{(2)^2+(2)^2}{4} - \frac{(5)^2+(3)^2}{8} - \frac{(1)^2+(5)^2}{6}$$

$$= 7.417,$$

$$k_2 = \frac{1}{2}\left[18 - \frac{(2)^2+(5)^2+(1)^2}{8} - \frac{(2)^2+(3)^2+(5)^2}{10}\right]$$

$$= 5.225$$

(c) Computing the variance component estimates.

The analysis of variance with the expectations of the mean squares is given below:

Source of Variation	d.f.	S.Sqs.	M.S.	E(MS)
Sires(S)	2	15.6786	7.8393	$\sigma_e^2+5.225\sigma_s^2$
Rations(R)	1	9.7036	9.7036	$\sigma_e^2+7.417\sigma_r^2$
Error	14	56.2964	4.0212	σ_e^2

$$\hat{\sigma}_r^2 = \frac{9.7036 - 4.0212}{7.417} = .766$$

$$\hat{\sigma}_s^2 = \frac{7.8393 - 4.0212}{5.225} = .731$$

8) Absorption of the $\mu + s_i$ equations.

(a) Calculation of the new or adjusted coefficients for the $\hat{r}_j$.

The calculations involved in the absorption of the $\mu + s_i$ equations may be obtained from the following table:

	$n_{i.}$	r_1	r_2	RHM
$\mu + s_1$:	4	2(.50000)	2(.50000)	16(4.00000)
$\mu + s_2$:	8	5(.62500)	3(.37500)	48(6.00000)
$\mu + s_3$:	6	1(.16667)	5(.83333)	30(5.00000)

The values in parentheses above are the numbers just outside the parentheses divided by $n_{i.}$, the diagonal coefficient for the $\mu + s_i$ equation. The calculation of adjustments which must be made in the coefficients for the $\hat{r}_j$ and in the RHM's for the r_j equations are now completed as follows:

$$C(r_1r_1) = 8 -(2)(.50000) -(5)(.62500) -(1)(.16667) = 3.70833.$$

$$C(r_1r_2) = C(r_2r_1)$$

$$= -(2)(.50000) -(5)(.37500) -(1)(.83333)$$

$$= -(.50000)(2) -(3)(.62500) -(5)(.16667) = -3.70833.$$

$$C(r_2r_2) = 10 -(2)(.50000) -(3)(.37500) -(5)(.83333) = 3.70835.$$

$$S(r_1) = 37 -(2)(4.00000) -(5)(6.00000) -(1)(5.00000)$$

$$= 37 -(.50000)(16) -(.62500)(48) -(.16667)(30) = -6.00000.$$

$$S(r_2) = 57 -(2)(4.00000) -(3)(6.00000) -(5)(5.00000) = 6.00000.$$

The reduced equations after absorption of the $\mu + s_i$ equations are as follows:

	$\hat{r}_1$	$\hat{r}_2$	RHM
r_1:	3.70833	-3.70833	-6.00000
r_2:	-3.70833	3.70835	6.00000

Except for rounding errors, the coefficients for $\hat{r}_j$ sum to zero by rows and columns and the RHM's sum to zero by column.

(b) Imposing the restriction that $\Sigma\, \hat{r}_j = 0$, computing the $\hat{r}_j$ and the inverse of the reduced matrix.j

The remaining equation after the subtraction by rows and columns is

$$14.83334\,\hat{r}_1 = -12.00000$$

$$\hat{r}_1 = \frac{-12.00000}{14.83334}$$

$$= -.8090$$

and

$$\hat{r}_2 = -\hat{r}_1 = .8090.$$

These are the same values obtained for the $\hat{r}_j$ when the solution to all equations was obtained directly.

The inverse of the reduced matrix, of only order one in this case, is

$$\frac{1}{14.83334} = .067416.$$

It will be noted that this is the same inverse element obtained for this position of the matrix inverse when the inversion of the μ, s_1, s_2 and r_1 reduced matrix was completed simultaneously.

(c) Back solution for the $\hat{\mu} + \hat{s}_i$ constants.

$$\begin{pmatrix} \hat{\mu} + \hat{s}_1 \\ \hat{\mu} + \hat{s}_2 \\ \hat{\mu} + \hat{s}_3 \end{pmatrix} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{6} \end{bmatrix} \left[\begin{pmatrix} 16 \\ 48 \\ 30 \end{pmatrix} - \begin{pmatrix} 2 & 2 \\ 5 & 3 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} -.8090 \\ .8090 \end{pmatrix} \right]$$

$$= \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 16.0000 \\ 49.6180 \\ 26.7640 \end{bmatrix}$$

$$= \begin{pmatrix} 4.0000 \\ 6.2022 \\ 4.4607 \end{pmatrix}.$$

(d) Computation of the variance-covariance inverse matrix and maximum likelihood estimates for the sire means.

$$G = -\begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ -4 \end{bmatrix} \begin{bmatrix} .067416 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -\frac{2}{8} \\ \frac{4}{6} \end{bmatrix} \begin{bmatrix} .067416 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -\frac{.134832}{8} \\ \frac{.269664}{6} \end{bmatrix} = \begin{bmatrix} 0 \\ -.016854 \\ .044944 \end{bmatrix}.$$

$$A = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{6} \end{bmatrix} \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ -4 \end{pmatrix} (0 \ -.016854 \ \ .044944) \right]$$

$$= \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 1.000000 & .000000 & .000000 \\ .000000 & 1.033708 & -.089888 \\ .000000 & -.067416 & 1.179776 \end{bmatrix}$$

$$= \begin{bmatrix} \underline{.250000} & .000000 & .000000 \\ .000000 & \underline{.129214} & -.011236 \\ .000000 & -.011236 & \underline{.196629} \end{bmatrix}$$

The A matrix inverse applies directly to the $\hat{\mu} + \hat{s}_i$. This matrix may be transformed, as shown below, to give the inverse elements which apply separately to $\hat{\mu}$ and the $\hat{s}_i$.

$$KAK' = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} \underline{.250000} & .000000 & .000000 \\ .000000 & \underline{.129214} & -.011236 \\ .000000 & -.011236 & \underline{.196629} \end{bmatrix} K'$$

$$= \frac{1}{3}\begin{bmatrix} .250000 & .117978 & .185393 \\ .500000 & -.117978 & -.185393 \\ -.250000 & .269664 & -.219101 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 2 \\ 1 & -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \underline{.061486} & .021848 & -.022160 \\ .021848 & \underline{.144819} & -.061174 \\ -.022160 & -.061174 & \underline{.112048} \end{bmatrix}$$

The transformed inverse elements for the r_1 column and row are obtained from

$$KG = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ -.016854 \\ -.044944 \end{bmatrix}$$

$$= \begin{bmatrix} .009363 \\ -.009363 \\ -.026217 \end{bmatrix}$$

The variance-covariance inverse obtained by this indirect manner is given in the following table:

	$\hat{\mu}$	$\hat{s}_1$	$\hat{s}_2$	$\hat{r}_1$
μ:	$\underline{.061486}$	.021848	-.022160	.009363
s_1:	.021848	$\underline{.144819}$	-.061174	-.009363
s_2:	-.022160	-.061174	$\underline{.112048}$	-.026217
r_1:	.009363	-.009363	-.026217	$\underline{.067416}$

It will noted that these inverse elements check within rounding error of those obtained directly. Of course, with only four equations this indirect method is inefficient. However, when there are a large number of constants to be fitted in one of the two sets, the absorption procedure is extremely useful. In many problems, the inverse for the section of the inverse matrix involving the constants absorbed is not required. However, the method of obtaining these inverse elements after absorption is useful in some problems, particularily where a large number of constants have been absorbed and tests of significance are desired among only a portion of those depending on the values obtained for $\hat{\mu} + \hat{s}_i$. Also, only the diagonal inverse elements for the $\hat{\mu} + \hat{s}_i$ can be computed by modifying this procedure slightly. These diagonal inverse elements are required to compute the approximate maximum likelihood estimates of the s_i effects. The $\hat{\mu} + \hat{s}_i$ for the numerical example are

$$\hat{\mu} + \hat{\hat{s}}_1 \doteq 4.8876 + \frac{.731}{.731 + (.250000)(4.0212)} (-.8876)$$

$$\doteq 4.8876 + (.4210)(-.8876)$$

$$\doteq 4.51$$

$$\hat{\mu} + \hat{\hat{s}}_2 \doteq 4.8876 + \frac{.731}{.731 + (.129214)(4.0212)}$$

$$\doteq 4.8876 + (.5845)(1.3146)$$

$$\doteq 5.65$$

$$\hat{\mu} + \hat{\hat{s}}_3 \doteq 4.8876 + \frac{.731}{.731 + (.196629)(4.0212)} (-.4270)$$

$$\doteq 4.8876 + (.4802)(-.4270)$$

$$\doteq 4.68$$

The "best" estimate of the expected performance of additional random progeny by each of these three sires, when an equal number are raised on each of these two rations, is given by the approximate maximum likelihood estimates above. The ranking of the $\hat{\mu} + \hat{\hat{s}}_i$ is very likely to be different than that of the $\hat{\mu} + \hat{s}_i$ when more numbers are available and when more variability in frequency occurs from subclass-to-subclass. Hence, the $\hat{\mu} + \hat{\hat{s}}_i$ are far more useful, than are the least-squares means, the $\hat{\mu} + \hat{s}_i$ in determining which animals are to be retained and which are to be culled in a selection program.

TWO-WAY CLASSIFICATION WITH INTERACTION

If data are classified in only two ways and it is necessary to consider the interaction effects, simplified methods are available for completing the least-squares analysis unless one or more of the subclasses are not filled. For the special case of only two classes of A or two classes of B a weighting method, described in several books on statistical methods, provides a short-cut procedure for computing all sums of squares in the analysis of variance. When there are more than two classes for both A and B and all subclasses are filled, the weighted squares of means method of analysis provides a short-cut procedure for computing the least-squares sums of squares for the main effects. In this case, a short-cut procedure is also available for computing the variance-covariance inverse. This procedure will be described and illustrated with a numerical example in this section.

When all AB subclasses are not filled and the interaction must be considered it is necessary to use least-squares procedures which involve matrix arithmetic in order to obtain unbiased estimates of the constants and the sums of squares for tests of significance. This direct procedure also is necessary if partial regression constants are being fitted along with those for μ, a_i, b_j, and the interaction constants, $(ab)_{ij}$, even though all subclasses are filled. In this latter case however, the alternative procedure for computing the inverse of the variance-covariance matrix that was presented in the section on the one-way classification with covariance may be useful.

The purpose of this section is to present the general least-squares analysis for data classified in two ways when constants for the interaction effects must also be fitted. Problems arising in the least-squares procedures when interaction constants are being fitted are presented in considerable detail.

Model

The general linear mathematical model for the two-way classification with interaction is

$$y_{ijk} = \mu + a_i + b_j + (ab)_{ij} + e_{ijk}$$

$$i = 1, 2, ---, p$$

$$j = 1, 2, ---, q$$

$$k = 1, 2, ---, n_{ij}$$

where:

y_{ijk} = the k^{th} observation in the j^{th} B class and i^{th} A class,

μ = the overall mean with equal subclass numbers,

a_i = effect of the i^{th} A class,

b_j = effect of the j^{th} B class,

$(ab)_{ij}$ = effect of the ij^{th} AB subclass after the average effects of A and B have been removed. These are the individual interaction effects expressed as a deviation from the mean μ.

e_{ijk} = random errors. Assumed to be NID$(0,\sigma_e^2)$.

The mathematical model is the same regardless of whether the a_i and/or the b_j are fixed or random effects. If both are fixed, the interaction effects are also fixed. When all effects except the errors are fixed the model is referred to as Model I of Eisenhart or the fixed model and if all effects except μ are random it is referred to as Model II of Eisenhart or the random model. The model is regarded as mixed when one set of effects (either the a_i or b_j) is fixed and the other is random. In the mixed model the interaction effects are random. The least-squares procedures required to compute the sum of squares for the analysis of variance are the same regardless of whether a set of effects is random or fixed. With fixed effects, the least-squares means with pairwise or orthogonal comparisons are often desired. On the other hand, with random effects, the investigator is interested in obtaining unbiased estimates of variance or covariance components. At least a portion of the matrix inverse to a variance-covariance matrix is required in order to test the significance of pairwise or orthogonal comparisons or to obtain estimates of variance or covariance components. Where feasible, i.e.,

if the number of equations is not too large, the inverse of the variance-covariance matrix for the complete model should be computed.

Least-Squares Equations

The least-squares equations for the two-way classification with interaction are given in tabular form below:

	$\hat{\mu}$	$\hat{a}_i$	$\hat{b}_j$	$(\hat{ab})_{ij}$	RHM
μ:	$n_{..}$	$n_{i.}$	$n_{.j}$	n_{ij}	$Y_{..}$
a_i:	$n_{i.}$	$0 \; n_{i.} \; 0$	n_{ij}	n_{ij}	$Y_{i.}$
b_j:	$n_{.j}$	n_{ij}	$0 \; n_{.j} \; 0$	n_{ij}	$Y_{.j}$
$(ab)_{ij}$:	n_{ij}	n_{ij}	n_{ij}	$0 \; n_{ij} \; 0$	Y_{ij}

This complete set of equations contains, (i) one equation for μ, (ii) one equation for each of the p classes of A, (iii) one equation for each of the q classes of B, and (iv) one equation for each of the subclasses of A and B which has one or more observations. If desirable, however, constants for only a selected group of the interaction effects need be fitted. In this case, equations for only the selected group of subclasses would be included for the interaction effects.

Imposing Restrictions on the Constants

A unique solution to the least-squares equations can not be obtained until they are reduced in number to the number of degrees of freedom. Although numerous restrictions may be imposed in order to accomplish this, the restriction that the constants for the main effects sum to zero within a set and that the constants for the $(ab)_{ij}$ sum to zero over each row and over each column is probably the most satisfactory. The linear mathematical model itself suggests this set of restrictions since the effects of a_i, b_j, $(ab)_{ij}$, and the e_{ij} are expressed as deviations from the mean μ.

Using the restriction which requires the setting of one of the constants in each set of main effects equal to zero (say, $\hat{a}_p$ and $\hat{b}_q$) suggests that the $(\hat{ab})_{iq}$, the $(\hat{ab})_{pj}$ and the $(\hat{ab})_{pq}$ interaction constants could also be set equal to zero, thereby, allowing the deletion of all these equations. Although a unique solution to the equations can be obtained when this is done, the estimates for the constants are entirely unsatisfactory, since they are

$$\hat{\mu}' = \hat{\mu} + \hat{a}_p + \hat{b}_q + \hat{ab}_{pq}$$

$$\hat{a}_i' = \hat{a}_i - \hat{a}_p + \hat{ab}_{iq} - \hat{ab}_{pq}$$

$$\hat{b}_j' = \hat{b}_j - \hat{b}_q + \hat{ab}_{pj} - \hat{ab}_{pq}$$

$$(\hat{ab})'_{ij} = (\hat{ab})_{ij} - (\hat{ab})_{iq} - (\hat{ab})_{pj} + (\hat{ab})_{pq}.$$

In addition to the confounding of the constant estimates when using these latter restrictions the total reduction in sum of squares obtained from

$$\sum_{i=1}^{p-1} \hat{a}'_i Y_{i.} + \sum_{j=1}^{q-1} \hat{b}'_j Y_{.j} + \sum_{i=1}^{p-1} \sum_{j=1}^{q-1} (\hat{ab})_{ij} Y_{ij}$$

is incorrect, being biased upwards. Also, the sums of squares for the various effects obtained from the sub-inverse matrices and the prime constant estimates are incorrect. Hence, the simple deletion of equations to remove dependencies among the least-squares equations when interaction constants are being fitted may lead to serious errors.

By imposing the restrictions that $\sum_i \hat{a}_i = \sum_j \hat{b}_j = \sum_i (\hat{ab})_{ij} = \sum_j (\hat{ab})_{ij} = 0$ on the prime estimates of the constants it is possible to compute the $\hat{a}_i$, $\hat{b}_j$ and $(\hat{ab})_{ij}$. In addition, the inverse matrix can be transformed to give the inverse that would have been obtained had the restrictions that $\sum_i \hat{a}_i = \sum_j \hat{b}_j = \sum_i (\hat{ab})_{ij} = \sum_j (\hat{ab})_{ij} = 0$ been imposed on the original least-squares equations. All these manipulations, however, require far more work than required in the subtractions and additions within the original equations to impose the restrictions that $\hat{a}_i$ and $\hat{b}_j$ sum to zero and the $(\hat{ab})_{ij}$ sum to zero by rows and columns.

When the restrictions that $\sum_i \hat{a}_i = \sum_j \hat{b}_j = \sum_i (\hat{ab})_{ij} = \sum_j (\hat{ab})_{ij} = 0$ are imposed on the least-squares equations it is necessary to carry out a number of subtractions and additions within the coefficient matrix and the right hand members. The subtractions required within the a_i and b_j equations are the same by column and by row as explained previously. Within the set of coefficients for the $(\hat{ab})_{ij}$ the subtractions and additions which may be conveniently chosen for each row are as follows:

$$n_{ij} - n_{iq} - n_{pj} + n_{pq}.$$

After these changes have been made by column the same procedure is followed by row with the modified coefficients for the $(ab)_{ij}$ equations and for the RHM's. Hence, the reduced RHM's for the remaining $(ab)_{ij}$ are

$$Y_{ij} - Y_{iq} - Y_{pj} + Y_{pq}.$$

Considerable manipulation of the coefficients and RHM's of the least-squares equations are required to impose these latter restrictions. However, if desirable, the subtractions and additions required are easily programmed on high speed computers and when the number of equations are small they can be completed quickly on a desk calculator or adding machine. By computing the subtractions and additions for all elements that are to remain in the reduced matrix a check on the computations is provided for the off-diagonal elements

of the coefficient matrix, since the reduced coefficient matrix must b symmetrical about the main diagonal.

Completing the Least-Squares Analysis

Any standard procedure may be used to compute the matrix inverse to the variance-covariance matrix and/or the solution to the reduced set of equations. It is often desirable to compute the solution of the equations directly for at least one RHM, and at the same time as the inverse matrix is being computed, in order to check on the accumulation of rounding errors. The difference between the constant estimates obtained from direct solution of the equations and those obtained by multiplying the inverse matrix by the reduced RHM's provides a measure of the accumulation of rounding errors. It is desirable to have the two estimates of the same constants agree to at least four significant digits. Checking of the off-diagonal elements of the inverse, when a method of inversion is used which computes all elements of the inverse, is an undesirable check on the accumulation of rounding errors since the rounding errors effect both off-diagonal elements in much the same manner. When no complete confounding of effects remains in the reduced least-squares matrix and no constants are being fitted for partial regressions, the accumulation of rounding errors is seldom a problem when standard procedures using seven or eight significant digits in all calculations are used to invert the matrix.

The inverse elements for the column and row for a_p (or b_q) may be obtained by adding the inverse elements for the a_i columns (or rows) and reversing the sign of the sum as explained in the first section. The same type of procedure is used to obtain the inverse elements for the interaction columns and rows which were eliminated, e.g.,

$$C^{\mu ab_{iq}} = C^{ab_{iq}\mu} = -\sum_{j=1}^{q-1} C^{\mu ab_{ij}}$$

$$C^{a_i ab_{iq}} = C^{ab_{iq} a_i} = -\sum_{j=1}^{q-1} C^{a_i ab_{ij}}$$

$$C^{b_j ab_{iq}} = C^{ab_{iq} b_j} = -\sum_{j=1}^{q-1} C^{b_j ab_{ij}}$$

$$C^{\mu ab_{pj}} = C^{ab_{pj}\mu} = -\sum_{i=1}^{p-1} C^{\mu ab_{ij}}$$

$$C^{\mu ab_{pq}} = C^{ab_{pq}\mu} = -\sum_{i=1}^{p-1} C^{\mu ab_{iq}} = -\sum_{j=1}^{q-1} C^{\mu ab_{pj}}$$

etc.

The sum of squares for error in the analysis of variance is computed from

$$\sum_{ijk}\sum\sum Y_{ijk}^2 - R\left[\mu, a_i, b_j, (ab)_{ij}\right],$$

where
$$R\left[\mu, a_i, b_j, (ab)_{ij}\right] = \hat{\mu}\, Y_{..} + \sum_{i=1}^{p-1} \hat{a}_i (Y_{i.} - Y_{p.}) + \sum_{j=1}^{q-1} \hat{b}_j (Y_{.j} - Y_{.q})$$

$$+ \sum_{i=1}^{p-1} \sum_{j=1}^{q-1} (\hat{ab})_{ij} (Y_{ij} - Y_{iq} - Y_{pj} + Y_{pq}).$$

The total reduction in sum of squares, $R\left[\mu, a_i, b_j, (ab)_{ij}\right]$, may also be computed from $\sum_{ij}\sum \frac{Y_{ij}^2}{n_{ij}}$ in this case, since all of the variability among the AB subclasses is accounted for by the constants being fitted. Hence, by computing this total reduction in both ways another check is provided on the accumulation of rounding errors during the solution of the equations.

The sums of squares for A, B, and AB may be computed by the direct procedure, $B'Z^{-1}B$, which involves the segments of the matrix inverse to the variance-covariance matrix and the constant estimates or from the indirect procedure involving differences in various reductions in sums of squares. If the indirect procedure is used the solution to other sets of equations is required. For example, the sum of squares for A (using the indirect method) is $R\left[\mu, a_i, b_j, (ab)_{ij}\right] - R\left[\mu, b_j, (ab)_{ij}\right]$. The $R\left[\mu, b_j, (ab)_{ij}\right]$ is obtained by deleting all equations for the a_i (by column and row) from the reduced set of least-squares equations, solving the remaining equations and computing the total reduction in the usual manner.

When the number of degrees of freedom for the interaction effects is large the sum of squares for interaction can usually be obtained more easily from $R\left[\mu, a_i, b_j, (ab)_{ij}\right] - R(\mu, a_i, b_j)$ than from $B'Z^{-1}B$, even though the complete inverse of the variance-covariance matrix is available. When the interaction constants must be fitted along with the main effects, the sums of squares for the main effects can usually be computed directly from $B'Z^{-1}B$ more easily than from differences in reductions. However, if all subclasses are filled the sums of squares for the main effects can be obtained with the weighted squares of means procedure more easily than with either of the methods considered above.

The least-squares means for the classes of A and the classes of B are $\hat{\mu} + \hat{a}_i$ and $\hat{\mu} + \hat{b}_j$, respectively. The standard errors for $\hat{\mu}$, $\hat{\mu} + \hat{a}_i$, and $\hat{\mu} + \hat{b}_j$ are computed in the same manner as described in the previous section from the appropriate inverse elements and the estimate of σ_e^2 from the error line of the analysis of variance. However, when the interaction effects are significant, the investigator is more interested in the AB subclass means rather than the class means. Since a least-squares subclass mean is

$$\hat{s}_{ij} = \hat{\mu} + \hat{a}_i + \hat{b}_j + (\hat{ab})_{ij},$$

the standard error may be computed from the inverse matrix and $\hat{\sigma}_e^2$ as follows.

$$s_{\hat{s}_{ij}} = \sqrt{(C^{\mu\mu} + C^{a_i a_i} + C^{b_j b_j} + C^{ab_{ij}} + 2C^{\mu a_i} + 2C^{\mu b_j} + 2C^{\mu ab_{ij}} + 2C^{a_i b_j} + 2C^{a_i ab_{ij}} + 2C^{b_i ab_{ij}})\, \hat{\sigma}_e^2}\,.$$

In the two-way classification when all subclasses are filled and the interaction constants are fitted, the standard error of the subclass mean, $\hat{s}_{ij}$, reduces to $\sqrt{\frac{1}{n_{ij}}\hat{\sigma}_e^2}$. The long formula for the standard error of the two-way subclass mean given above is useful when all cells are not filled or when all interaction constants are not fitted as well as when other constants, such as partial regressions or other main effects, must also be fitted at the same time.

Mean separation procedures can be completed for the $\hat{a}_i$ and the $\hat{b}_j$ in the same manner as described in previous sections. Procedures for obtaining sums of squares for individual degree-of-freedom-orthogonal comparisons among the $\hat{a}_i$ or among the $\hat{b}_j$ are also explained in previous sections. The same procedures may be used to partition all of the variation among the $\hat{a}_i$, the $\hat{b}_j$ and that due to the $\widehat{ab}_{ij}$ into single degree-of-freedom-orthogonal contrasts. In this case, the diagonal inverse matrix involving the s_{ij} would be transformed with the appropriate transformation matrix. For example, suppose there are two classes of A and three classes of B and the following single degree of freedom orthogonal contrasts are desired among the subclass means:

	$\hat{s}_{11}$	$\hat{s}_{12}$	$\hat{s}_{13}$	$\hat{s}_{21}$	$\hat{s}_{22}$	$\hat{s}_{23}$
μ	1	1	1	1	1	1
A	1	1	1	-1	-1	-1
B	2	-1	-1	2	-1	-1
	0	1	-1	0	1	-1
AB	2	-1	-1	-2	1	1
	0	1	-1	0	-1	1

Using the rules discussed in a previous section, the transformation matrix, K, is

$$K = \frac{1}{24}\begin{bmatrix} 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & -4 & -4 & -4 \\ 6 & -3 & -3 & 6 & -3 & -3 \\ 0 & 6 & -6 & 0 & 6 & -6 \\ 6 & -3 & -3 & -6 & 3 & 3 \\ 0 & 6 & -6 & 0 & -6 & 6 \end{bmatrix}$$

The transformed matrix, T, is then computed from

$$T = KD^{-1}K'$$

and the constants for the orthogonal contrasts, $\hat{c}_i$, from $KB = \hat{c}_i$, where B is a column vector of the $\hat{s}_{ij}$.

The "within" subclass mean square is the appropriate error term for testing the significance of the interaction regardless of whether the model is fixed, mixed or random. Likewise, this error mean square is the appropriate error term for testing the significance of both main effects in the fixed model and for the random main effects in the mixed model. **There is no exact test** of significance for the fixed main effects in the mixed model or for both main effects in the random model when both unequal subclass frequencies and the interaction exist.

The expectation of the mean squares for the random model is as follows:

	E(MS)
A	$\sigma_e^2 + k_4\sigma_{ab}^2 + k_5\sigma_a^2$
B	$\sigma_e^2 + k_2\sigma_{ab}^2 + k_3\sigma_b^2$
AB	$\sigma_e^2 + k_1\sigma_{ab}^2$
Error	σ_e^2

Since k_1, k_2 and k_4 are all different values when unequal subclass numbers exist (except when $p=q=2$) even though all subclasses are filled, no exact test of significance exists for either the a_i or b_j effects in the random model. However, this is not a serious problem since the investigator is primarily interested in obtaining the best unbiased estimates of the variance components when all effects, except μ, are random rather than in tests of significance. He will often have a priori evidence that the a_i and b_j effects exist.

The coefficients for σ_{ab}^2 in the mean square for AB, k_1, and the coefficients for σ_b^2 and σ_a^2, k_3 and k_5, may be computed by either the direct or the indirect procedure described in the previous section. The other coefficients, k_2 and k_4, are computed by the indirect procedure. If every AB subclass is filled all of these coefficients, except k_1, can be computed from the weights used in the weighted squares of means procedure as described in the next sub-section.

The expectation of the mean squares for the mixed model with the a_i effects random and the b_j effects fixed is as follows:

	E(MS)
A	$\sigma_e^2 + k_5\sigma_a^2$
B	$\sigma_e^2 + k_2\sigma_{ab}^2 + k_3\sigma_b^2$
AB	$\sigma_e^2 + k_1\sigma_{ab}^2$
Error	σ_e^2

The variance due to the a_i, σ_a^2, is now estimated over a fixed set of B effects and therefore the interaction variance which gets into the sum of squares for A can not be separated from the main effects of A. Hence, the denominator

mean square for testing the significance of the A effects (over this set of fixed B classes) is the error mean square. Also, the estimate of σ_a^2 is obtained by computing only k_5 in the mixed model.

Since k_1 and k_2 will differ when unequal subclass frequencies exist no exact test of significance exist for the b_j effects in either the random or mixed model. The approximate test, in which the mean square for B is divided by the mean square for AB, decreases in desirability as the amount of unequal frequencies increase and as the size of σ_{ab}^2 increases.

The expectation of the mean squares for the fixed model is identical with that given above for the mixed model except that $k_2\sigma_{ab}^2$ disappears from the E(MS) for B. Hence, all effects are tested for significance by using the error mean square as the denominator for F.

With the fixed model and missing subclasses, it must be realized that all estimates of constants and all tests of significance resulting from the least-squares analysis are relative to the population of subclasses actually included in the data. Generalization of the results to combinations of A and B which do not occur in the data, of course, can not be made. This would be true even though the interaction effects for the subclasses available did not exist since no estimate of the interaction effects outside of the data can be obtained.

When data are analyzed under the mixed model, the absence of AB subclasses from the data creates a very undesirable situation as far as estimation of variance components is concerned. In this case, the variance due to the random effects, say σ_a^2, is estimated over a fixed set of effects and the absence of AB subclasses means that some of the a_i effects would not be measured over all of the B fixed classes. Hence, it is usually desirable to eliminate the A classes which are not observed with all B classes (when the a_i are random and the b_j are fixed) or eliminate the B classes which are not observed in combination with all A classes if the estimation of σ_a^2 is of primary concern. If the primary interest in the analysis is to obtain unbiased estimates of $\mu + b_j$ with minimum error when missing subclasses exist, there is no need to eliminate any of the filled subclasses available since the effects of B are measured over a random sample of a_i effects.

Short-cut Procedures

When data are classified in only two ways with all subclasses filled and the interaction must be considered, the weighted squares of means procedure can be used to compute the sums of squares for the main effects which are adjusted for the interaction effects. A short-cut procedure can also be used to compute the coefficients of variance components in the expectation of the mean squares for the main effects. However, the weighted squares of means procedure does not provide for the computation of the sum of squares for interaction (AB) or a short-cut procedure for computing the coefficient for σ_{ab}^2, k_1, in the mean square of AB. The interaction sum of squares is usually computed by the general indirect procedure for computing a

least-squares sum of squares in the two-way classification, i.e.,

$$\text{S.Sqs.--AB} = R\left[\mu, a_i, b_j, (ab)_{ij}\right] - R(\mu, a_i, b_j)$$

$$= \sum_i \sum_j \frac{Y_{ij}^2}{n_{ij}} - \hat{\mu}\, Y_{\cdot\cdot} - \sum_{i=1}^{p-1} \hat{a}_i (Y_{i\cdot} - Y_{p\cdot}) - \sum_{j=1}^{q-1} \hat{b}_j (Y_{\cdot j} - Y_{\cdot q})$$

where $\hat{\mu}$, the $\hat{a}_i$ and the $\hat{b}_j$ are the constants which solve the reduced least-squares equations when the interaction effects are omitted from the model. If k_1 is required it can usually best be computed by the general indirect procedure for computing coefficients of variance components, i.e

$$k_1 = \frac{1}{r-p-q+1}\left(n_{\cdot\cdot} - \sum_i \sum_j R^{ij} N_{ij}\right)$$

where r is the number of AB subclasses; the R^{ij} are the inverse elements of the matrix inverse to the reduced-least-squares-coefficient matrix when the interaction constants are omitted from the model, the N_{ij} are elements of the "associated" sums matrix computed from NN' as explained in the previous section.

The weighted squares of means procedure for computing estimates of the $\mu + a_i$ and $\mu + b_j$ the least-squares sums of squares for A and B and coefficients of variance components in the E(MS) is given below:

$$\hat{\mu} + \hat{a}_i = \frac{\sum_j \hat{s}_{ij}}{q} \qquad w_i = \frac{q^2}{\sum_j 1/n_{ij}}$$

$$\hat{\mu} + \hat{b}_j = \frac{\sum_i \hat{s}_{ij}}{p} \qquad v_j = \frac{p^2}{\sum_i 1/n_{ij}}$$

$$\text{S.Sqs.--A} = \sum_i w_i\, \hat{a}_i^2 - \frac{\left(\sum_i w_i \hat{a}_i\right)^2}{w_\cdot}$$

$$\text{S.Sqs.--B} = \sum_j v_j \hat{b}_j^2 - \frac{\left(\sum_j v_j \hat{b}_j\right)^2}{v_\cdot}$$

$$k_2 = \frac{1}{p(q-1)}\left(v_\cdot - \frac{\sum_j v_j^2}{v_\cdot}\right)$$

$$k_3 = \frac{1}{q-1}\left(v_\cdot - \frac{\sum_j v_j^2}{v_\cdot}\right)$$

$$k_4 = \frac{1}{q(p-1)} \left(w_. - \frac{\sum_i w_i^2}{w_.}\right)$$

$$k_5 = \frac{1}{p-1} \left(w_. - \frac{\sum_i w_i^2}{w_.}\right) .$$

When all subclasses are filled, standard errors of the $\hat{a}_i$ and $\hat{b}_j$ may be computed directly from the weighting factors and the estimate of the error variance as follows:

$$s_{\hat{a}_i} = \sqrt{\frac{1}{q^2/\sum_j 1/n_{ij}} \hat{\sigma}_e^2} = \sqrt{\frac{1}{w_i} \hat{\sigma}_e^2}$$

$$s_{\hat{b}_j} = \sqrt{\frac{1}{p^2/\sum_i 1/n_{ij}} \hat{\sigma}_e^2} = \sqrt{\frac{1}{v_j} \hat{\sigma}_e^2}$$

The standard error of a subclass mean, $\hat{s}_{ij}$, is

$$\sqrt{\frac{1}{n_{ij}} \hat{\sigma}_e^2}$$

and the standard error of the difference between any two subclass means, say, $\hat{s}_{ij} - \hat{s}_{i'j.}$, is

$$\sqrt{\left(\frac{1}{n_{ij}} + \frac{1}{n_{i'j}}\right)\hat{\sigma}_e^2} .$$

Hence, mean separation procedures can be applied easily to the subclass means.

If mean separation procedures are to be applied to the $\hat{\mu} + \hat{a}_i$, or the $\hat{\mu} + \hat{b}_j$, or if single degree-of-freedom-orthogonal comparisons are desired, inverse elements from the matrix inverse to the variance-covariance matrix are required. When all cells are filled the complete or partial matrix inverse can be obtained by a short-cut procedure. In effect, the diagonal inverse matrix for the subclasses, D^{-1}, is transformed with the use of the appropriate transformation matrix, K, to obtain C^{-1}, the matrix inverse to the complete variance-covariance matrix, i.e., $C^{-1} = KD^{-1}K'$. The transformation matrix, K, is the functional relationship matrix which gives the association of the $\hat{s}_{ij}$, the subclass means, and $\hat{\mu}$, the $\hat{a}_i$, the $\hat{b}_j$, and $(\hat{ab})_{ij}$. For example, if p = 2 and q = 3 the transformation matrix is

$$K = \frac{1}{6}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 2 & -1 & -1 & 2 & -1 & -1 \\ -1 & 2 & -1 & -1 & 2 & -1 \\ 2 & -1 & -1 & -2 & 1 & 1 \\ -1 & 2 & -1 & 1 & -2 & 1 \end{bmatrix}$$

where the row coefficients refer to the μ, a_i, b_j, and $(ab)_{ij}$ constants in order and the column coefficients refer to the subclass means arranged in the usual order, $\hat{s}_{11}$, $\hat{s}_{12}$, $\hat{s}_{13}$, $\hat{s}_{21}$, $\hat{s}_{22}$, and $\hat{s}_{23}$. The formulas which give the relationship of the subclass means and the constants when $\sum_i \hat{a}_i = \sum_j \hat{b}_j = \sum_i (ab)_{ij} = \sum_j (\hat{ab})_{ij} = 0$ are as follows:

$$\hat{\mu} = \frac{1}{pq} \sum_{ij}\sum \hat{s}_{ij}$$

$$\hat{a}_i = \frac{p-1}{pq} \sum_j \hat{s}_{ij} - \frac{1}{pq} \sum_{i'} \sum_j \hat{s}_{i'j}$$

$$\hat{b}_j = \frac{q-1}{pq} \sum_i \hat{s}_{ij} - \frac{1}{pq} \sum_{ij'}\sum \hat{s}_{ij'}$$

$$(\hat{ab})_{ij} = \frac{(p-1)(q-1)}{pq} \hat{s}_{ij} - \frac{p-1}{pq} \sum_{j'} \hat{s}_{ij'} - \frac{q-1}{pq} \sum_{i'} \hat{s}_{i'j} + \frac{1}{pq} \sum_{i'} \sum_{j'} \hat{s}_{i'j'}$$

The coefficients for the $\hat{s}_{ij}$ in these formulas are the elements of the transformation matrix (K). Coefficients for the transformation matrix are easily written and could probably be generated by a high speed computer for almost any combination of p and q. The numerator coefficients for any $(ab)_{ij}$ row can quickly be obtained from the products of the numerator coefficients in the a_i and b_j rows.

Since D^{-1} is a diagonal matrix the computations can be arranged in a systematic manner, as illustrated in the section on the one-way classification with covariance, and only those inverse elements required need be computed. After computing the inverse elements, mean separation procedures and computations required for testing the significance of orthogonal contrasts are carried out as previously described.

Absorption of the $\mu + a_i$ Equations

When subclasses are missing in the two way classification with interaction or if the analysis includes the fitting of all constants for the two way classification with interaction plus constants for other effects, such as main effects or partial regressions, absorption of the $\mu + a_i$ equations into the equations for the remaining constants may be desirable. However, since constants are being fitted here for the $(ab)_{ij}$ effects the proportionate reduction in size of the matrix is not as great as is likely to be the case when interaction constants are not being fitted. If the absorption process is to be utilized the model should usually be arranged so that $p > q$.

The new coefficients in the equations for the b_j and the $(ab)_{ij}$ after absorption of the $\mu + a_i$ equations are computed as follows:

$$C(b_j b_j) = n_{.j} - \sum_i \frac{n_{ij}^2}{n_{i.}}$$

$$C(b_j b_{j'}) = -\sum_i \frac{n_{ij} n_{ij'}}{n_{i.}}$$

$$C\left[b_j (ab)_{ij}\right] = C\left[(ab)_{ij}(ab)_{ij}\right] = n_{ij} - \frac{n_{ij}^2}{n_{i.}}$$

$$C\left[b_j (ab)_{ij'}\right] = C\left[(ab)_{ij}(ab)_{ij'}\right] = -\frac{n_{ij} n_{ij'}}{n_{i.}}$$

$$C\left[(ab)_{ij}(ab)_{i'j}\right] = C\left[(ab)_{ij}(ab)_{i'j'}\right] = 0$$

The new right hand members for the b_j and $(ab)_{ij}$ equations are

$$S(b_j) = Y_{.j} - \sum_i \frac{n_{ij} Y_{i.}}{n_{i.}}$$

$$S\,(ab)_{ij} = Y_{ij} - \frac{n_{ij} Y_{ij}}{n_{i.}} \, .$$

After absorption of the $\mu + a_i$ equations, restrictions may be imposed that $\sum_j \hat{b}_j = \sum_i (\hat{ab})_{ij} = \sum_j (\hat{ab})_{ij} = 0$ and the necessary subtractions completed to remove the dependencies prior to solution and/or inversion. With estimates for the $\hat{b}_j$ and the $(\hat{ab})_{ij}$ the original equations for $\mu + a_i$ can be solved to obtain the $\hat{\mu} + \hat{a}_i$. With the segment of the matrix inverse for the b_j and $(ab)_{ij}$, the inverse elements for the remaining segments of the complete matrix inverse may be computed, if desired, with matrix multiplication procedures. The basic procedure for completing these calculations was presented in the previous section on the two-way classification without interaction and will be given in general matrix notation in the next section.

Estimation of Variance Components

When all effects in the model are random, except μ, two useful short-cut methods which provide unbiased estimates of the variance components will be considered. Both of these methods are due to Henderson (1953)[6] and are referred to by him as Method 1 and Method 3. Both methods are based on least-squares principles. However, the estimates obtained for the variance

6/ Henderson, C. R. Ibid.

components by the two methods will differ for any given set of data and both sets of estimates will differ from those obtained from the complete least-squares analysis as described above. The variances of estimates of variance components from data with unequal subclass frequencies are unknown.

However, intuitively it seems that Method 1 of Henderson provides the least efficient estimates and the complete least-squares analysis provides the most efficient estimates. Hence, where feasible, the variance component estimates should be obtained from the complete least-squares analysis. In many instances, however, the number of equations is too large to attempt the complete least-squares analysis when the interaction must be considered and all cells are not filled.

Method 1 of Henderson requires the least labor of all three methods. Sums of squares for the two main effects, the interaction and the within subclass are computed in the usual manner for balanced designs except that appropriate divisors are used for the unequal numbers. Coefficients for the variance components in the mean squares (or sums of squares) are then computed and the computed mean squares are set equal to their expectation. The resulting set of simultaneous equations is solved to obtain estimates of the variance components. In the two-way classification with interaction the analysis is as follows:

Source of Variation	d.f.	S.Sqs	E(MS)
A	p-1	$\sum_i \frac{Y_{i.}^2}{n_{i.}} - \frac{Y_{..}^2}{n_{..}}$	$\sigma_e^2+k_7\sigma_{ab}^2+k_8\sigma_b^2+k_9\sigma_a^2$
B	q-1	$\sum_j \frac{Y_{.j}^2}{n_{.j}} - \frac{Y_{..}^2}{n_{..}}$	$\sigma_e^2+k_4\sigma_{ab}^2+k_5\sigma_b^2+k_6\sigma_a^2$
AB	r-p-q+1	$\sum_{ij}\sum \frac{Y_{ij}^2}{n_{ij}} - \sum_i \frac{Y_{i.}^2}{n_{i.}} - \sum_j \frac{Y_{.j}^2}{n_{.j}} + \frac{Y_{..}^2}{n_{..}}$	$\sigma_e^2+k_1\sigma_{ab}^2+k_2\sigma_b^2+k_3\sigma_a^2$
Error	n..-r	$\sum_{ijk}\sum\sum Y_{ijk}^2 - \sum_{ij}\sum \frac{Y_{ij}^2}{n_{ij}}$	σ_e^2

The nine k values in E(MS) are computed as follows:

$$k_1 = \frac{1}{r-p-q+1}\left(n_{..} - \sum_i \frac{\sum_j n_{ij}^2}{n_{i.}} - \sum_j \frac{\sum_i n_{ij}^2}{n_{.j}} + \frac{\sum_{ij}\sum n_{ij}^2}{n_{..}}\right)$$

$$k_2 = \frac{1}{r-p-q+1}\left(\frac{\sum_j n_{.j}^2}{n_{..}} - \sum_i \frac{\sum_j n_{ij}^2}{n_{i.}}\right)$$

$$k_3 = \frac{1}{r-p-q+1}\left(\frac{\sum_i n_{i.}^2}{n_{..}} - \sum_j \frac{\sum_i n_{ij}^2}{n_{.j}}\right)$$

$$k_4 = \frac{1}{q-1}\left(\sum_j \frac{\sum_i n_{ij}^2}{n_{.j}} - \frac{\sum_i\sum_j n_{ij}^2}{n_{..}}\right)$$

$$k_5 = \frac{1}{q-1}\left(n_{..} - \frac{\sum_j n_{.j}^2}{n_{..}}\right)$$

$$k_6 = \frac{1}{q-1}\left(\sum_j \frac{\sum_i n_{ij}^2}{n_{.j}} - \frac{\sum_i n_{i.}^2}{n_{..}}\right)$$

$$k_7 = \frac{1}{p-1}\left(\sum_i \frac{\sum_j n_{ij}^2}{n_{i.}} - \frac{\sum_i\sum_j n_{ij}^2}{n_{..}}\right)$$

$$k_8 = \frac{1}{p-1}\left(\sum_i \frac{\sum_j n_{ij}^2}{n_{i.}} - \frac{\sum_j n_{.j}^2}{n_{..}}\right)$$

$$k_9 = \frac{1}{p-1}\left(n_{..} - \frac{\sum_i n_{i.}^2}{n_{..}}\right)$$

Many terms are common among these formulas for the k's. Actually, there are only five terms involving the n_{ij}, $n_{i.}$, $n_{.j}$, and $n_{..}$ that must be computed from the data.

Method 3 of Henderson is a least-squares analysis in which the sums of squares and expectations of the error and AB interaction mean squares are computed in the usual manner. The sums of squares for the main effects, A and B, are computed from

$$\text{S.Sqs.--A} = R(\mu,a_i,b_j) - R(\mu,b_j)$$

$$= R(\mu,a_i,b_j) - \sum_j \frac{Y_{.j}^2}{n_{.j}}$$

and

$$\text{S.Sqs.--B} = R(\mu,a_i,b_j) - (\mu,a_i)$$

$$= R(\mu,a_i,b_j) - \sum_i \frac{Y_{i.}^2}{n_{i.}}$$

rather than

$$\text{S.Sqs.---A} = R\left[\mu,a_i,b_j,(ab)_{ij}\right] - R\left[\mu,b_j,(ab)_{ij}\right]$$

$$\text{S.Sqs.--B} = R\left[\mu, a_i, b_j, (ab)_{ij}\right] - R\left[\mu, a_i, (ab)_{ij}\right]$$

as is done when the complete least-squares analysis (or weighted squares of means analysis, when applicable) is made. The E(MS) in the analysis of variance can then be set up as follows for the random model:

	E(MS)
A	$\sigma_e^2 + k_4'\sigma_{ab}^2 + k_5'\sigma_a^2$
B	$\sigma_e^2 + k_2'\sigma_{ab}^2 + k_3'\sigma_b^2$
AB	$\sigma_e^2 + k_1\sigma_{ab}^2$
Error	σ_e^2

The coefficient for σ_{ab}^2 in E(MS) for AB is obtained in the usual manner for a least-squares analysis, as previously described. The coefficients for σ_b^2 and σ_a^2, k_3' and k_5', may be computed directly from the inverse of the square symmetrical segments of the matrix inverse to the variance-covariance matrix when the interaction effects have been deleted from the model. However, k_3' and k_5' can more easily be computed by the indirect method since the last reduction in sum of squares used in computing both A and B reduces to a "between" sum of squares. The formulas for computing k_3' and k_5' therefore become

$$k_3' = \frac{1}{q-1}\left(n_{..} - \sum_i \frac{\sum_j n_{ij}^2}{n_{i.}}\right)$$

$$k_5' = \frac{1}{p-1}\left(n_{..} - \sum_j \frac{\sum_i n_{ij}^2}{n_{.j}}\right) .$$

The coefficients for σ_{ab}^2 in E(MS) for A and B, k_2' and k_4', are computed by the indirect method as follows:

$$k_2' = \frac{1}{q-1}\left(\sum_i\sum_j R^{ij}N_{ij} - \sum_i \frac{\sum_j n_{ij}^2}{n_{i.}}\right)$$

$$k_4' = \frac{1}{p-1}\left(\sum_i\sum_j R^{ij}N_{ij} - \sum_j \frac{\sum_i n_{ij}^2}{n_{.j}}\right)$$

where the R^{ij} are the inverse elements from the matrix inverse to the variance-covariance matrix for the model, $y_{ijk} = \mu + a_i + b_j + e_{ijk}$, and N_{ij} are the elements of the "associated sums" matrix computed from NN'.

Numerical Example

The hypothetical set of data analyzed in the three previous sections to illustrate computational procedures will now be analyzed under the two-way classification model with interaction. The general least-squares procedures that would be required with missing subclasses will first be given for these data followed by the weighted squares of means procedures that are applicable when all cells are filled. The mathematical model for this analysis is

$$y_{ijk} = \mu + s_i + r_j + (sr)_{ij} + e_{ijk}$$

$$i = 1, 2, 3$$

$$j = 1, 2$$

$$k = 1, 2, \ldots, n_{ij}$$

where

y_{ijk} = the gain of the k^{th} barrow on the j^{th} ration by the i^{th} sire,

μ = the overall population mean with equal subclass frequencies,

s_i = effect of the i^{th} sire,

r_j = effect of the j^{th} ration,

$(sr)_{ij}$ = interaction effects,

e_{ijk} = random errors.

General Least-Squares Analysis

1) Least-squares equations

The least-squares equations for these data are represented in tabular form below:

	$\hat{\mu}$	$\hat{s}_1$	$\hat{s}_2$	$\hat{s}_3$	$\hat{r}_1$	$\hat{r}_2$	$(\hat{sr})_{11}$	$(\hat{sr})_{12}$	$(\hat{sr})_{21}$	$(\hat{sr})_{22}$	$(\hat{sr})_{31}$	$(\hat{sr})_{32}$	RHM
μ:	18	4	8	6	8	10	2	2	5	3	1	5	94
s_1:	4	4	0	0	2	2	2	2	0	0	0	0	16
s_2:	8	0	8	0	5	3	0	0	5	3	0	0	48
s_3:	6	0	0	6	1	5	0	0	0	0	1	5	30
r_1:	8	2	5	1	8	0	2	0	5	0	1	0	37
r_2:	10	2	3	5	0	10	0	2	0	3	0	5	57
$(sr)_{11}$:	2	2	0	0	2	0	2	0	0	0	0	0	11
$(sr)_{12}$:	2	2	0	0	0	2	0	2	0	0	0	0	5
$(sr)_{21}$:	5	0	5	0	5	0	0	0	5	0	0	0	23
$(sr)_{22}$:	3	0	3	0	0	3	0	0	0	3	0	0	25
$(sr)_{31}$:	1	0	0	1	1	0	0	0	0	0	1	0	3
$(sr)_{32}$:	5	0	0	5	0	5	0	0	0	0	0	5	27

2) Imposing the restrictions that $\sum_i \hat{s}_i = \sum_j \hat{r}_j = \sum_i (\hat{sr})_{ij} = \sum_j (\hat{sr})_{ij} = 0$

When these restrictions or constraints are imposed on the constant estimates and the appropriate subtractions and additions are made the reduced least-squares equations appear as follows when placed in tabular form:

	$\hat{\mu}$	$\hat{s}_1$	$\hat{s}_2$	$\hat{r}_1$	$(\hat{sr})_{11}$	$(\hat{sr})_{21}$	RHM
μ:	18	-2	2	-2	4	6	94
s_1:	-2	10	6	4	-4	-4	-14
s_2:	2	6	14	6	-4	-2	18
r_1:	-2	4	6	18	-2	2	-20
$(sr)_{11}$:	4	-4	-4	-2	10	6	30
$(sr)_{21}$:	6	-4	-2	2	6	14	22

3) Matrix inverse to the variance-covariance matrix and constant estimates.

The inverse of the 6x6 symmetrical coefficient (or variance-covariance) matrix is given in the table below:

	$\hat{\mu}$	$\hat{s}_1$	$\hat{s}_2$	$\hat{r}_1$	$(\hat{sr})_{11}$	$(\hat{sr})_{21}$
μ:	.075926	.007407	-.031481	.018519	-.018519	-.029630
s_1:		.159259	-.051852	-.018519	.018519	.029630
s_2:			.120370	-.029630	.029630	.007407
r_1:				.075926	.007407	-.031481
$(sr)_{11}$:					.159259	-.051852
$(sr)_{21}$:						.120370

The inverse elements to the left of the main diagonal are omitted from the table since they would be the same as those to the right of the main diagonal. Several of the off-diagonal values in the matrix inverse are of the same magnitude although in some cases they differ in sign. Also, the sub-matrix of inverse elements for the interaction constants is the same as the sub-matrix for the sire constants. These similarities occurred primarily because of only two classes for rations in this set of data and will not occur when p and q are each larger than three.

The constant estimates obtained by multiplying the inverse matrix and the RHM's in the reduced matrix together are as follows:

$$\hat{\mu} = 4.8889 \qquad \hat{r}_1 = -.5222$$
$$\hat{s}_1 = -.8889 \qquad (\hat{sr})_{11} = 2.0222$$
$$\hat{s}_2 = 1.5778 \qquad (\hat{sr})_{21} = -1.3444$$

Estimates of the remaining constants are computed as follows:

$$\hat{s}_3 = -(-.8889 + 1.5778) = -.6889$$
$$\hat{r}_2 = -(-.5222) = .5222$$
$$(\hat{sr})_{12} = -(2.0222) = -2.0222$$
$$(\hat{sr})_{22} = -(-1.3444) = 1.3444$$
$$(\hat{sr})_{31} = -(2.0222 - 1.3444) = -.6778$$
$$(\hat{sr})_{32} = -(-2.0222 + 1.3444) = .6778$$

Four decimal digits are carried in the constant estimates to avoid serious rounding errors in the computation of sums of squares. Obviously, these estimates are not accurate to four decimals.

4) Sums of squares for the analysis of variance.

$$\begin{aligned}\text{S.Sqs.---Error} &= \sum_{ijk}\sum\sum y_{ijk}^2 - R\left[\mu, s_i, r_j, (sr)_{ij}\right] \\ &= 568 - (4.8889)(94) - (-.8889)(-14) \\ &\quad -(1.5778)(18) - (-.5222)(-20) - (2.0222)(30) \\ &\quad -(-1.3444)(22) \\ &= 26.0652\end{aligned}$$

When constants are fitted for all degrees of freedom among the subclasses the error sum of squares may also be computed from

$$\sum_{ijk}\sum\sum y_{ijk}^2 - \sum_{ij}\sum \frac{Y_{ij}^2}{n_{ij}}$$

With this example,

$$\text{S.Sqs.---Error} = 568 - 541.9333 = 26.0667$$

which checks (within rounding error) with the sum of squares obtained for error using the general computing method above. It will be noted that this is the same sum of squares that was used for error in the two-way classification without interaction for testing the significance of the interaction effects.

The sums of squares for sires (S), rations (R) and the interaction (SR) are computed by the general direct procedure ($B'Z^{-1}B$) as follows:

$$\begin{aligned}\text{S.Sqs.---S} &= \begin{bmatrix}-.8889 & 1.5778\end{bmatrix}\begin{bmatrix}.159259 & -.051852 \\ -.051852 & .120370\end{bmatrix}^{-1}\begin{bmatrix}-.8889 \\ 1.5778\end{bmatrix} \\ &= \begin{bmatrix}-.8889 & 1.5778\end{bmatrix}\begin{bmatrix}7.303395 & 3.146096 \\ 3.146096 & 9.662967\end{bmatrix}\begin{bmatrix}-.8889 \\ 1.5778\end{bmatrix} \\ &= \begin{bmatrix}-1.5281 & 12.4497\end{bmatrix}\begin{bmatrix}-.8889 \\ 1.5778\end{bmatrix}\end{aligned}$$

$$= 21.0015$$

$$\text{S.Sqs.---R} = \frac{(-.5222)^2}{.075926} = 3.5916$$

$$\text{S.Sqs.---SR} = \begin{bmatrix} 2.0222 & -1.3444 \end{bmatrix} \begin{bmatrix} .159259 & -.051852 \\ -.051852 & .120370 \end{bmatrix}^{-1} \begin{bmatrix} 2.0222 \\ -1.3444 \end{bmatrix}$$

$$= \begin{bmatrix} 10.5393 & -6.6289 \end{bmatrix} \begin{bmatrix} 2.0222 \\ -1.3444 \end{bmatrix}$$

$$= 30.2245$$

Except for rounding errors, it will be noted that the sum of squares obtained for interaction in this manner is the same as that computed in the last section from

$$\sum_i \sum_j \frac{Y_{ij.}^2}{n_{ij}} - R(\mu, s_i, r_j) \ .$$

However, it should also be noted that the sums of squares for S and R above are considerably different from those obtained for these sources of variation when the interaction effects were disregarded. Also, of course, the constant estimates for the s_i and r_j are different in the two analyses. Still another point worth noting in the two analyses is the difference in the diagonal inverse elements for μ, s_1, s_2, and r_1. When adjustment must be made for the interaction effects these inverse elements are always larger unless equal subclass frequencies exist. The only factor causing these inverse elements to be larger and the constant estimates and the sums of squares for S and R to differ is the unequal subclass frequencies. The distribution, and not the magnitude of the interaction effects, is the only factor involved in making these adjustments.

When both the s_i and the r_j are fixed the analysis of variance may be completed as follows:

Source of Variation	d.f.	S.Sqs.	M.S.	F
S	2	21.0015	10.5008	4.83*
R	1	3.5916	3.5916	1.65 n.s.
SR	2	30.2245	15.1122	6.96*
Error	12	26.0652	2.1721	

*Indicates significance at the .05 level of probability.

5) Least-squares means, standard errors and individual comparisons.

The least-squares means, which may be of interest in an analysis such as this, are

$\hat{\mu} = 4.9$ - - - The overall mean expected with equal numbers.

$\hat{\mu} + \hat{s}_1 = 4.0$ - - - Sire No. 1 mean.

$\hat{\mu} + \hat{s}_2 = 6.5$ - - - Sire No. 2 mean.

$$\hat{\mu} + \hat{s}_3 = 4.2 \text{ - - - Sire No. 3 mean.}$$
$$\hat{\mu} + \hat{r}_1 = 4.4 \text{ - - - Ration No. 1 mean.}$$
$$\hat{\mu} + \hat{r}_2 = 5.4 \text{ - - - Ration No. 2 mean.}$$
$$\hat{\mu} + \hat{s}_1 + \hat{r}_1 + (\hat{sr})_{11} = 5.5 \text{ - - - Sire 1 - Ration 1 subclass mean.}$$
$$\hat{\mu} + \hat{s}_1 + \hat{r}_2 + (\hat{sr})_{12} = 2.5 \text{ - - - Sire 1 - Ration 2 subclass mean.}$$
$$\hat{\mu} + \hat{s}_2 + \hat{r}_1 + (\hat{sr})_{21} = 4.6 \text{ - - - Sire 2 - Ration 1 subclass mean.}$$
$$\hat{\mu} + \hat{s}_2 + \hat{r}_2 + (\hat{sr})_{22} = 8.3 \text{ - - - Sire 2 - Ration 2 subclass mean.}$$
$$\hat{\mu} + \hat{s}_3 + \hat{r}_1 + (\hat{sr})_{31} = 3.0 \text{ - - - Sire 3 - Ration 1 subclass mean.}$$
$$\hat{\mu} + \hat{s}_3 + \hat{r}_2 + (\hat{sr})_{32} = 5.4 \text{ - - - Sire 3 - Ration 2 subclass mean.}$$

The standard error for each of these means may be computed from the elements of the inverse matrix and the estimate of σ_e^2, 2.1721, from the analysis of variance in the usual manner which has previously been illustrated. Pairwise tests of significance among the sire means or among the subclass means may be completed with the "t" test or Duncan's Multiple Range Test with the information now available as have also previously been illustrated. The method of computing sums of squares for single degree-of-freedom-orthogonal contrasts was also described and illustrated in the earlier sections.

6) Estimation of variance components.

Although the number of degrees of freedom for sires and rations are wholly inadequate to obtain accurate estimates of variance components for these two main effects or for the interaction, the computational procedures required to obtain these estimates will be illustrated with this small set of data.

If the effects of sires and rations are both regarded as random,7/ the expectations of the mean squares in the analysis of variance are as follows:

	d.f.	E(MS)
S	p-1	$\sigma_e^2 + k_4\sigma_{sr}^2 + k_5\sigma_s^2$
R	q-1	$\sigma_e^2 + k_2\sigma_{sr}^2 + k_3\sigma_r^2$
SR	r-p-q+1	$\sigma_e^2 + k_1\sigma_{sr}^2$
Error	n..-r	σ_e^2

The coefficient for σ_{sr}^2 in the SR mean square, k_1, and the coefficients for σ_r^2, and σ_s^2, k_3 and k_5, may be computed by the direct method

7/ The effects of rations would probably never be regarded as random in practice. This is done here merely to illustrate variance component estimation with the random model.

from the Z^{-1} matrices that were used to compute the sums of squares. These computations are shown below:

$$k_1 = \frac{7.303395 + 9.662967 - (\frac{2}{2})(3.146096)}{6}$$

$$= \frac{13.820266}{6} = 2.303$$

$$k_3 = \frac{1/.075926}{2} = \frac{13.17072}{2} = 6.585$$

$$k_5 = \frac{7.303395 + 9.662967 - (\frac{2}{2})(3.146096)}{3}$$

$$= \frac{13.820266}{3} = 4.607$$

The coefficients for σ^2_{sr} in the E(MS) for R and S are computed by the indirect method as follows:

$$k_2 = \frac{1}{I}\,(18 - \sum_{ij}\sum R^{ij} N_{ij})$$

$$R = \begin{bmatrix} \underline{18} & -2 & 2 & 4 & 6 \\ -2 & \underline{10} & 6 & -4 & -4 \\ 2 & 6 & \underline{14} & -4 & -2 \\ 4 & -4 & -4 & \underline{10} & 6 \\ 6 & -4 & -2 & 6 & \underline{14} \end{bmatrix}$$

$$R^{-1} = \begin{bmatrix} \underline{.071409} & .011924 & -.024254 & -.020326 & -.021952 \\ & \underline{.154742} & -.059079 & .020326 & .021952 \\ & & \underline{.108807} & .032521 & -.004878 \\ & & & \underline{.158536} & .048781 \\ & & & & \underline{.107317} \end{bmatrix}$$

when the off-diagonals on the left are omitted.

$$N_{ij} = NN' = \begin{bmatrix} 2 & 2 & 5 & 3 & 1 & 5 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix} N'$$

$$
\begin{array}{c}
\begin{array}{cccccccccc}
\mu & s_1 & s_2 & s_3 & (sr)_{11} & (sr)_{12} & (sr)_{21} & (sr)_{22} & (sr)_{31} & (sr)_{32}
\end{array}\\
= \begin{bmatrix}
\underline{68} & 8 & 34 & 26 & 4 & 4 & 25 & 9 & 1 & 25 \\
 & \underline{8} & 0 & 0 & 4 & 4 & 0 & 0 & 0 & 0 \\
 & & \underline{34} & 0 & 0 & 0 & 25 & 9 & 0 & 0 \\
 & & & \underline{26} & 0 & 0 & 0 & 0 & 1 & 25 \\
 & & & & \underline{4} & 0 & 0 & 0 & 0 & 0 \\
 & & & & & \underline{4} & 0 & 0 & 0 & 0 \\
 & & & & & & \underline{25} & 0 & 0 & 0 \\
 & & & & & & & \underline{9} & 0 & 0 \\
 & & & & & & & & \underline{1} & 0 \\
 & & & & & & & & & \underline{25}
\end{bmatrix}
\end{array}
$$

when the off-diagonals on the left are omitted.

The reduced N_{ij} matrix, after subtracting by rows and columns, is

$$
\begin{bmatrix}
\underline{68} & -18 & 8 & 24 & 40 \\
 & \underline{34} & 26 & -24 & -24 \\
 & & \underline{60} & -24 & -8 \\
 & & & \underline{34} & 26 \\
 & & & & \underline{60}
\end{bmatrix}
$$

This matrix may be obtained more easily from the N' matrix where 0, 1 and -1 codes are used.

when the left half is again omitted.

Hence,

$$k_2 = \left[18 - (68)(.071409) - (-18)(.011924) - \cdots - (60)(.107317)\right]$$
$$= 18 - 15.804544 = 2.195$$

The calculations involved in computing k_4 by the indirect method are as follows:

$$k_4 = \frac{1}{2}\left(18 - \sum_{ij}\sum R^{ij}N_{ij}\right)$$

$$
R = \begin{bmatrix}
\underline{18} & -2 & 4 & 6 \\
-2 & \underline{18} & -2 & 2 \\
4 & -2 & \underline{10} & 6 \\
6 & 2 & 6 & \underline{14}
\end{bmatrix}
$$

$$
R^{-1} = \begin{bmatrix}
\underline{.067416} & .009363 & -.009363 & -.026217 \\
 & \underline{.061486} & .021848 & -.022160 \\
 & & \underline{.144819} & -.061174 \\
 & & & \underline{.112047}
\end{bmatrix}
$$

$$N_{ij} = NN' = \begin{bmatrix} 2 & 2 & 5 & 3 & 1 & 5 \\ 2 & 0 & 5 & 0 & 1 & 0 \\ 0 & 2 & 0 & 3 & 0 & 5 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix} N'$$

$$\begin{array}{c} \begin{array}{ccccccccc} \mu & r_1 & r_2 & (sr)_{11} & (sr)_{12} & (sr)_{21} & (sr)_{22} & (sr)_{31} & (sr)_{32} \end{array} \\ = \begin{bmatrix} \underline{68} & 30 & 38 & 4 & 4 & 25 & 9 & 1 & 25 \\ & \underline{30} & 0 & 4 & 0 & 25 & 0 & 1 & 0 \\ & & \underline{38} & 0 & 4 & 0 & 9 & 0 & 25 \\ & & & \underline{4} & 0 & 0 & 0 & 0 & 0 \\ & & & & \underline{4} & 0 & 0 & 0 & 0 \\ & & & & & \underline{25} & 0 & 0 & 0 \\ & & & & & & \underline{9} & 0 & 0 \\ & & & & & & & \underline{1} & 0 \\ & & & & & & & & \underline{25} \end{bmatrix} \end{array}$$

The reduced N_{ij} matrix, after subtracting by rows and columns, is

$$\begin{bmatrix} \underline{68} & -8 & 24 & 40 \\ & \underline{68} & -18 & 8 \\ & & \underline{34} & 26 \\ & & & \underline{60} \end{bmatrix}$$

Hence,

$$k_4 = \frac{1}{2}(18 - 13.393274) = \frac{1}{2}(4.606726) = 2.303$$

Use of the indirect method to compute k_2 and k_4 involves a great deal of arithmetic even when the number of classes for both A and B is small. A shorter and more direct method should be available to compute these coefficients. Perhaps one will be discovered in the near future. Until then, the use of the complete least-squares analysis for estimating variance components when interactions must be considered and subclasses are missing will, no doubt, be limited.

Estimates of the variance components may now be computed from the complete least-squares analysis as follows:

Source of Variation	d.f.	M.S.	E(MS)
S	2	10.5008	$\sigma_e^2 + 2.303\ \sigma_{sr}^2 + 4.607\ \sigma_s^2$
R	1	3.5916	$\sigma^2 + 2.195\ \sigma_{sr}^2 + 6.585\ \sigma_r^2$
SR	2	15.1122	$\sigma_e^2 + 2.303\ \sigma_{sr}^2$
Error	12	2.1721	σ_e^2

$$\hat{\sigma}_e^2 = 2.172 \qquad \hat{\sigma}_r^2 = -1.657$$

$$\hat{\sigma}_{sr}^2 = 5.618 \qquad \hat{\sigma}_s^2 = -1.001$$

The variance of these variance component estimates, of course, is extremely large. Many more degrees of freedom are needed for both S and R in order to obtain reliable estimates of the variance components. The variance component analysis for these data is presented merely to illustrate the computational procedures.

Weighted Squares of Means Analysis

When all degrees of freedom among a set of subclasses are partitioned into a set of orthogonal comparisons, as is done in factorial analyses, the estimates of all constants are determined from the subclass means. In this case, weights are easily determined that are useful for computing sums of squares, variances of constant estimates and coefficients of variance components. Steps required in the application of the weighted squares of means procedures to the analysis of the data in the numerical example will now be given.

(1) Computation of constant estimates.

$$\hat{\mu} + \hat{s}_i = \frac{\sum_j \hat{s}_{ij}}{q}$$

$$\hat{\mu} + \hat{s}_1 = \frac{5.5+2.5}{2} = 4.0000$$

$$\hat{\mu} + \hat{s}_2 = \frac{4.6000+8.3333}{2} = 6.4667$$

$$\hat{\mu} + \hat{s}_3 = \frac{3.0+5.4}{2} = 4.2000$$

$$\hat{\mu} + \hat{r}_j = \frac{\sum_i \hat{s}_{ij}}{p}$$

$$\hat{\mu} + \hat{r}_1 = \frac{5.5+4.6+3.0}{3} = 4.3667$$

$$\hat{\mu} + \hat{r}_2 = \frac{2.5000+8.3333+5.4000}{3} = 5.4111$$

$$\hat{\mu} = \frac{4.0000+6.46667+4.2000}{3} = \frac{4.3667+5.4111}{2} = 4.8889$$

$$\hat{s}_1 = 4.0000 - 4.8889 = -.8889$$

$$\hat{s}_2 = 6.4667 - 4.8889 = 1.5778$$

$$\hat{s}_3 = 4.2000 - 4.8889 = -.6889$$

$$\hat{r}_1 = 4.3667 - 4.8889 = -.5222$$

$$\hat{r}_2 = 5.4111 - 4.8889 = .5222$$

(2) Computation of weights.

$$w_i = \frac{q^2}{\sum_j 1/n_{ij}} \qquad v_j = \frac{p^2}{\sum_i 1/n_{ij}}$$

$$w_1 = \frac{4}{\frac{1}{2}+\frac{1}{2}} = 4.0000 \qquad v_1 = \frac{9}{\frac{1}{2}+\frac{1}{5}+1} = 5.2941$$

$$w_2 = \frac{4}{\frac{1}{5}+\frac{1}{3}} = 7.5000 \qquad v_2 = \frac{9}{\frac{1}{2}+\frac{1}{3}+\frac{1}{5}} = 8.7097$$

$$w_3 = \frac{4}{1+\frac{1}{5}} = 3.3333$$

(3) Computation of sums of squares.

S.Sqs. - - - Error = Within Subclass S.Sqs. = 26.0067

S.Sqs. - - - SR = 30.2245

The sum of squares for interaction must be computed from the least-squares analysis, i.e., $B'Z^{-1}B$ or $R[\mu, a_i, b_j, (ab)_{ij}] - R(\mu, a_i, b_j)$.

$$\text{S.Sqs.---S} = (4.0000)(-.8889)^2 + (7.5000)(1.5778)^2 + (3.3333)(-.6889)^2 - \frac{[(4.0000)(-.8889) + (7.5000)(1.5778) + (3.3333)(-.6889)]^2}{4.0000 + 7.5000 + 3.3333}$$

$$= 21.0013$$

$$\text{S.Sqs.---R} = (5.2941)(-.5222)^2 + (8.7097)(.5222)^2 - \frac{[(5.2941)(-.5222) + (8.7097)(.5222)]^2}{5.2941 + 8.7097}$$

$$= 3.5916$$

It will be noted that these sums of squares for R and S agree within rounding errors with those obtained by the general least-squares procedure, $B'Z^{-1}B$.

(4) Computation of k's.

As pointed out previously, k_1 must be computed by the general

least-squares procedure even though all subclasses are filled. All other k's may be computed from the weights as follows:

$$k_2 = \frac{1}{(3)(1)}\left[(5.2941+8.7097) - \frac{(5.2941)^2+(8.7097)^2}{5.2941+8.7097}\right]$$

$$= \frac{1}{3}\,(14.0038 - \frac{103.8864}{14.0038})$$

$$= \frac{1}{3}\,(6.5854) = 2.195$$

$$k_3 = \frac{1}{1}\,(6.5854) = 6.585$$

$$k_4 = \frac{1}{(2)(2)}\left[14.8333 - \frac{(4.0000)^2+(7.5000)^2+(3.3333)^2}{14.8333}\right]$$

$$= \frac{1}{4}\,(9.2135) = 2.303$$

$$k_5 = \frac{1}{2}\,(9.2135) = 4.607$$

It will be noted that these k values (k_2, k_3, k_4, and k_5) computed in this manner agree with those obtained by the general least-squares procedure. When all subclasses are filled

$$k_2 = \frac{1}{p}\,k_3$$

and

$$k_4 = \frac{1}{q}\,k_5\ .$$

Short-cut Procedure of Computing the Matrix Inverse

When all degrees of freedom among a set of subclasses is partitioned into a set of orthogonal comparisons, the complete matrix inverse to the variance-covariance matrix can be computed from

$$C^{-1} = KD^{-1}K'$$

where K is the transformation matrix and D^{-1} is the diagonal inverse matrix for the set of subclasses. In the present problem these computations are simplified with the aid of the following table:

		μ	s_1	s_2	r_1	$(sr)_{11}$	$(sr)_{21}$	Mean	D^{-1}	Constant Estimates
s_{11}		1	2	-1	1	2	-1	5.5000	.500000	4.8889
s_{12}		1	2	-1	-1	-2	1	2.5000	.500000	-.8889
s_{21}	$(\frac{1}{6})$	1	-1	2	1	-1	2	4.6000	.200000	1.5778
s_{22}		1	-1	2	-1	1	-2	8.3333	.333333	-.5222
s_{31}		1	-1	-1	1	-1	-1	3.0000	1.000000	2.0222
s_{32}		1	-1	-1	-1	1	1	5.4000	.200000	-1.3444

The transpose of the transformation matrix (K') appears on the left in the table. The values in the D^{-1} column are the reciprocals of the subclass numbers. The constant estimates are computed from

$$\hat{\mu} = \frac{1}{6}\,(5.5000 + 2.5000 \;+ + +\; 5.4000) = 4.8889$$

$$\hat{s}_1 = \frac{1}{6}\left[(2)(5.5000) + (2)(2.5000) - (1)(4.6000 \;---\; (1)(5.4000)\right]$$

$$= -.8889$$

etc.

The inverse elements of the matrix inverse to the variance-covariance matrix may be computed one at a time as follows:

$$c^{\mu\mu} = \frac{1}{36}\left[(1)^2(.500000) + (1)^2(.500000) + + + (1)^2(.200000)\right]$$

$$= .075926$$

$$c^{\mu s_1} = \frac{1}{36}\left[(1)(2)(.500000) + (1)(2)(.500000) + + + (1)(-1)(.200000)\right]$$

$$= .007407$$

$$c^{\mu s_2} = \frac{1}{36}\left[(1)(-1)(.500000) + (1)(-1)(.500000) + + + (1)(-1)(.200000)\right]$$

$$= -.031481$$

-

-

$$c^{sr_{21}sr_{21}} = \frac{1}{36}\left[(-1)^2(.500000) + (1)^2(.500000) + + + (1)^2(.200000)\right]$$

$$= .120370$$

Application of Method 1 of Henderson

When this method is used to obtain estimates of the variance components, the sums of squares for the analysis of variance are computed as follows:

$$\text{S.Sqs.---S} = \frac{(16)^2}{4} + \frac{(48)^2}{8} + \frac{(30)^2}{6} - \frac{(94)^2}{18}$$

$$= 502 - 490.8889 = 11.1111$$

$$\text{S.Sqs.---R} = \frac{(37)^2}{8} + \frac{(57)^2}{10} - \frac{(94)^2}{18}$$

$$= 496.0250 - 490.8889 = 5.1361$$

$$\text{S.Sqs.---SR} = \frac{(11)^2}{2} + \frac{(5)^2}{2} + + + \frac{(27)^2}{5} - 502 - 496.0250 + 490.8889$$

$$= 541.9333 - 507.1361 = 34.7972$$

$$\text{S.Sqs.--Error} = 568 - 541.9333 = 26.0667$$

The analysis of variance is as follows:

Source of Variation	d.f.	S.Sqs.	M.S.	E(MS)
S	2	11.1111	5.5556	$\sigma_e^2 + k_7\sigma_{sr}^2 + k_8\sigma_r^2 + k_9\sigma_s^2$
R	1	5.1361	5.1361	$\sigma_e^2 + k_4\sigma_{sr}^2 + k_5\sigma_r^2 + k_6\sigma_s^2$
SR	2	34.7972	17.3986	$\sigma_e^2 + k_1\sigma_{sr}^2 + k_2\sigma_r^2 + k_3\sigma_s^2$
Error	12	26.0667	2.1722	σ_e^2

The terms required for the computation of the k values for E(MS) are computed as follows:

$$\sum_i \frac{\sum_j n_{ij}^2}{n_{i.}} = \frac{(2)^2+(2)^2}{4} + \frac{(5)^2+(3)^2}{8} + \frac{(1)^2+(5)^2}{6} = 10.5833$$

$$\sum_j \frac{\sum_i n_{ij}^2}{n_{.j}} = \frac{(2)^2+(5)^2+(1)^2}{8} + \frac{(2)^2+(3)^2+(5)^2}{10} = 7.5500$$

$$\frac{\sum_i n_{i.}^2}{n_{..}} = \frac{(4)^2+(8)^2+(6)^2}{18} = 6.4444$$

$$\frac{\sum_j n_{.j}^2}{n_{..}} = \frac{(8)^2+(10)^2}{18} = 9.1111$$

$$\frac{\sum_{ij}\sum n_{ij}^2}{n_{..}} = \frac{(2)^2+(2)^2+(5)^2+(3)^2+(1)^2+(5)^2}{18} = 3.7778$$

The nine k's are now computed as follows:

$$k_1 = \frac{1}{2}\,(18 - 10.5833 - 7.5500 + 3.7778) = 1.8222$$

$$k_2 = \frac{1}{2}\,(9.1111 - 10.5833) = -.7361$$

$$k_3 = \frac{1}{2}\,(6.4444 - 7.5500) = -.5528$$

$$k_4 = \frac{1}{1}\,(7.5500 - 3.7778) = 3.7722$$

$$k_5 = \frac{1}{1}\,(18 - 9.1111) = 8.8889$$

$$k_6 = \frac{1}{1}\,(7.5500 - 6.4444) = 1.1056$$

$$k_7 = \frac{1}{2}\,(10.5833 - 3.7778) = 3.4028$$

$$k_8 = \frac{1}{2}\,(10.5833 - 9.1111) = .7361$$

$$k_9 = \frac{1}{2}\,(18 - 6.4444) = 5.7778$$

When the computed mean squares are set equal to the expected mean squares and the resulting equations are solved, the following estimates for the variance components are obtained:

$$\hat{\sigma}_e^2 = 2.172 \qquad \hat{\sigma}_r^2 = -2.087$$

$$\hat{\sigma}_{sr}^2 = 6.593 \qquad \hat{\sigma}_s^2 = -3.031$$

The estimates of σ_{sr}^2, σ_r^2 and σ_s^2 do not agree with those obtained from the complete least-squares analysis. Although these estimates probably have larger sampling errors than those obtained with the complete least-squares analysis they are unbiased provided all effects in the model, except μ, can be regarded as random.

Application of Method 3 of Henderson

With this method the sums of squares for error and the interaction remain the same as with the complete least-squares analysis. Also the coefficient for σ_{sr}^2 in the expectation of the interaction mean square, k_1, remains the same. The sums of squares for S and R are computed as follows under this method:

$$\begin{aligned} \text{S.Sqs.---S} &= R(\mu, s_i, r_j) - R(\mu, r_j) \\ &= 511.7036 - 496.0250 = 15.6786 \end{aligned}$$

$$S.Sqs.—R = R(\mu, s_j, r_j) - R(\mu, s_j)$$

$$= 511.7036 - 502.0000 = 9.7036$$

It should be noted that these are the same sums of squares that were computed for S and R when the interaction was assumed to be non-existent. The coefficients for σ_s^2 and σ_r^2 in the expectation of the mean squares for S and R (k_5' and k_3') respectively, also remain the same as when interaction was disregarded. The coefficients for σ_{sr}^2 in the expectations of the mean squares for S and R, k_4' and k_2' are computed as follows:

$$k_2' = \frac{1}{1}(\sum_{ij}\sum R^{ij}N_{ij} - \sum_i \frac{\sum_j n_{ij}^2}{n_{i.}})$$

$$k_4' = \frac{1}{2}(\sum_{ij}\sum R^{ij}N_{ij} - \sum_j \frac{\sum_i n_{ij}^2}{n_{.j}})$$

$$R = \begin{bmatrix} \underline{18} & -2 & 2 & -2 \\ & \underline{10} & 6 & 4 \\ & & \underline{14} & 6 \\ & & & \underline{18} \end{bmatrix}$$

$$R^{-1} = \begin{bmatrix} \underline{.061486} & .021848 & -.022160 & .009363 \\ & \underline{.144819} & -.061174 & -.009363 \\ & & \underline{.112047} & -.026217 \\ & & & \underline{.067416} \end{bmatrix}$$

$$N_{ij} = NN' = \begin{bmatrix} 2 & 2 & 5 & 3 & 1 & 5 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 2 & 0 & 5 & 0 & 1 & 0 \\ 0 & 2 & 0 & 3 & 0 & 5 \end{bmatrix} N'$$

$$\begin{bmatrix} \underline{68} & 8 & 34 & 26 & 30 & 38 \\ & \underline{8} & 0 & 0 & 4 & 4 \\ & & \underline{34} & 0 & 25 & 9 \\ & & & \underline{26} & 1 & 25 \\ & & & & \underline{30} & 0 \\ & & & & & \underline{38} \end{bmatrix}$$

When the appropriate subtractions and additions are made by column and row N reduces to:

$$
\begin{bmatrix}
\underline{68} & -18 & 8 & -8 \\
 & \underline{34} & 26 & 24 \\
 & & \underline{60} & 40 \\
 & & & \underline{68}
\end{bmatrix}
$$

Hence,

$$k_2' = 13.3933 - 10.5833 = 2.810$$

$$k_4' = \frac{1}{2}\,(13.3933 - 7.5500) = 2.922$$

The analysis of variance with the expectations of the mean squares can now be set up as follows:

Source of Variation	d.f.	S.Sqs	M.S.	E(MS)
S	2	15.6786	7.8393	$\sigma_e^2 + 2.922\ \sigma_{sr}^2 + 5.225\ \sigma_s^2$
R	1	9.7036	9.7036	$\sigma_e^2 + 2.810\ \sigma_{sr}^2 + 7.417\ \sigma_r^2$
SR	2	30.2297	15.1148	$\sigma_e^2 + 2.303\ \sigma_{sr}^2$
Error	12	26.0667	2.1722	σ_e^2

Estimates of the variance components when computed by Method 3 of Henderson are as follows:

$$\sigma_e^2 = 2.172 \qquad \sigma_r^2 = -1.114$$

$$\sigma_{sr}^2 = 5.620 \qquad \sigma_s^2 = -2.058$$

Three methods of computing unbiased estimates of variance components under the random model have been illustrated with this small numerical example. Presumably, estimates of variance components obtained from a complete least-squares analysis have smaller sampling errors than those obtained with either Methods 1 or 3 of Henderson. However, short-cut procedures are still needed before either the complete least-squares analysis or Method 3 of Henderson will be feasible for many sets of data.

MULTIPLE AND NESTED CLASSIFICATIONS

Most sets of animal breeding data with disproportionate subclass numbers are classified in more than two-ways. Often, this is also true with other types of data where unequal subclass numbers exist. Many of the computational procedures presented in the previous sections are directly applicable to any least-squares analysis. On the other hand, special techniques are often required to complete least-squares analyses which involve multiple classifications and the combination of multiple and nested classifications. In this section will be presented some of these special techniques and short-cuts and a computational example which includes multiple classifications, a nested classification, interaction, and two partial regressions in one analysis. In addition, the general least-squares analysis is presented in matrix notation.

Models

Although there are many specific designs for which the linear mathematical models are standard, the most appropriate for use in the analysis of any given set of data may differ considerably from the standard models. Since the accurateness and validity of conclusions drawn from any analysis are highly dependent upon how accurately the selected mathematical model describes the biology involved, the importance of selecting the appropriate model can not be overemphasized. On the other hand, it is doubtful that all assumptions necessary for tests of significance and estimation procedures to be strictly valid, from a mathematical viewpoint, will ever be entirely satisfied. Nevertheless, every effort should be made to select the model which does allow the assumptions required, with respect to homogeneity of variance and independence of errors, to be most nearly satisfied. At the same time, however, the model should not be so complex that the analysis can not be completed at a reasonable cost.

When unequal subclass numbers exist and constants are fitted for a set of effects (either main effects, interaction effects or regression) which really has no influence on the variability in y, the constants for other sets of effects that are estimated at the same time and the tests of significance for such effects, although unbiased, may be very inefficient. Hence, in the selection of the model for the analysis it is important to exclude from the model sets of effects which are known not to effect the variability. At the same time, however, it is important to include in the model all effects which really do effect the variability of y in order that the estimates obtained in the analysis will be unbiased. This places the investigator in somewhat of a quandary. A good rule to follow is to place a set of effects in the model when some doubt exists concerning whether such effects are really zero or not. Methods are available (and will be presented in this section) for deleting a set of effects after it has been found that the constants for the effects within the set probably equal zero.

The specific model for consideration at present is

$$y_{ijkl} = \alpha + a_i + b_{ij} + c_k + (ac)_{ik} + dD_{ijkl} + gG_{ijkl} + e_{ijkl} \quad \text{---} \quad (5)$$

$i = 1, 2, \ldots, p \qquad l = 1, 2, \ldots, n_{ijk}$

$j = 1, 2, \ldots, q_i \qquad s$ = No. of AB subclasses

$k = 1, 2, \ldots, r \qquad t$ = No. of AC subclasses

where:

y_{ijkl} = the l^{th} observation in the k^{th} C class and in the j^{th} B class within the i^{th} A class.

α = the population mean when equal frequencies exist in all subclasses and $D_{ijkl} = G_{ijkl} = 0$.

a_i = effect of the i^{th} A class.

b_{ij} = effect of the j^{th} B class within the i^{th} A class.

c_k = effect of the k^{th} C class.

d = partial regression of the y_{ijkl} on the D_{ijkl}.

D_{ijkl} = an independent continuous variable.

g = partial regression of the y_{ijkl} on the G_{ijkl}.

G_{ijkl} = a second independent continuous variable.

e_{ijkl} = random errors which are assumed to be $NID(0,\sigma_e^2)$. The normality assumption is required to validate tests of significance.

The population mean when equal frequencies exist is

$$\mu = \alpha + d\,\bar{\bar{D}} + g\,\bar{\bar{G}}$$

where $\bar{\bar{D}}$ and $\bar{\bar{G}}$ are the means of the D_{ijkl} and the G_{ijkl}. However, the population mean for any combination of values for D and G can be computed if α, d, and g are known.

The model (5) does not include a term for the interaction of the b_{ij} with the c_k, $(bc)_{ijk}$, which could exist. It is assumed here that a priori information clearly indicates that the $(bc)_{ijk}$ effects equal zero. However, in the completion of the least-squares analysis under model (5) a procedure will be given for testing the significance of the $(bc)_{ijk}$ interaction effects. If the a priori information was incorrect and the interaction (BC:A) is significant the estimates of constants for the other effects are biased.

The procedures to be followed in the analysis of a set of data under model (5) depend on several factors. If all effects in the model, except the random errors, are regarded as fixed a complete least-squares analysis is preferred. In this case, the inverse of the complete variance-covariance matrix would be required if individual comparisons are to be made among least-squares means. Also, if all effects are fixed the number of independent least-squares equations is not likely to be large and therefore, the complete analysis is not too difficult to accomplish. However, the effects for a nested classification, such as b_{ij}, usually represent a random sample of those effects and not a selected group. In this case, the number of equations is likely to be large and direct inversion of all independent equations may be impractical. The mere fact that a large number of original least-squares equations exist does not prohibit a least-squares analysis from being completed. Short-cut methods are available which allow the investigator to obtain unbiased estimates, and in many cases the most efficient estimates, of all constants and most tests of significance desired even though the number of equations is large.

Oftentimes when model (5) is appropriate both p and r are small but the q_i are large. When this is true the equations for $\alpha + a_i + b_{ij}$ should first be absorbed into the equations for c_k, $(ac)_{ik}$, d, and g even though the inverse of the complete variance-covariance matrix is desired. In effect, the inverse of the complete coefficient matrix is then obtained more easily by a matrix-partitioning procedure than if the complete matrix inverse is obtained by direct inversion. In order to present these procedures (as well as others) in general terms which apply to any least-squares analysis, model (5) will be given in vector notation and then associated with general matrix notation. Model (5) in vector notation may be written as follows:

$$y = \alpha + A + B + C + AC + d\,D + g\,G + e$$

where

$$y = \begin{pmatrix} y_{1111} \\ y_{1112} \\ \cdot \\ \cdot \\ \cdot \\ y_{pqrn_{pqr}} \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha \\ \cdot \\ \cdot \\ \cdot \\ \alpha \end{pmatrix} + \begin{pmatrix} a_1 \\ a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_p \end{pmatrix} + \begin{pmatrix} b_{11} \\ b_{11} \\ \cdot \\ \cdot \\ \cdot \\ b_{pq} \end{pmatrix} + \begin{pmatrix} c_1 \\ c_1 \\ \cdot \\ \cdot \\ \cdot \\ c_r \end{pmatrix} + \begin{pmatrix} (ac)_{11} \\ (ac)_{11} \\ \cdot \\ \cdot \\ \cdot \\ (ac)_{pr} \end{pmatrix} + \begin{pmatrix} dD_{1111} \\ dD_{1112} \\ \cdot \\ \cdot \\ \cdot \\ dD_{pqrn_{pqr}} \end{pmatrix}$$

$$+ \begin{pmatrix} gG_{1111} \\ gG_{1112} \\ \cdot \\ \cdot \\ \cdot \\ gG_{pqrn_{pqr}} \end{pmatrix} + \begin{pmatrix} e_{1111} \\ e_{1112} \\ \cdot \\ \cdot \\ \cdot \\ e_{pqrn_{pqr}} \end{pmatrix}$$

A general form for any model is

$$y = XB_1 + ZB_2 + e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

If $p = 2$, $q_i = 3$ for all i, $r = 2$ and $n_{ijk} = 1$ in all cases for an analysis under model (5)

$$X = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$B_1 = \begin{pmatrix} \alpha + a_1 + b_{11} \\ \alpha + a_1 + b_{12} \\ \alpha + a_1 + b_{13} \\ \alpha + a_2 + b_{21} \\ \alpha + a_2 + b_{22} \\ \alpha + a_2 + b_{23} \end{pmatrix}$$

when the constants for α, a_i, and b_{ij} are combined into a subclass constant. This would be necessary in order to absorb all equations for the $\alpha+a_i+b_{ij}$. Also,

$$Z = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & D_{111} & G_{111} \\ 0 & 1 & 0 & 1 & 0 & 0 & D_{112} & G_{112} \\ 1 & 0 & 1 & 0 & 0 & 0 & D_{121} & G_{121} \\ 0 & 1 & 0 & 1 & 0 & 0 & D_{122} & G_{122} \\ 1 & 0 & 1 & 0 & 0 & 0 & D_{131} & G_{131} \\ 0 & 1 & 0 & 1 & 0 & 0 & D_{132} & G_{132} \\ 1 & 0 & 0 & 0 & 1 & 0 & D_{211} & G_{211} \\ 0 & 1 & 0 & 0 & 0 & 1 & D_{212} & G_{212} \\ 1 & 0 & 0 & 0 & 1 & 0 & D_{221} & G_{221} \\ 0 & 1 & 0 & 0 & 0 & 1 & D_{222} & G_{222} \\ 1 & 0 & 0 & 0 & 1 & 0 & D_{231} & G_{231} \\ 0 & 1 & 0 & 0 & 0 & 1 & D_{232} & G_{232} \end{pmatrix}$$

$$B_2 = \begin{pmatrix} c_1 \\ c_2 \\ (ac)_{11} \\ (ac)_{12} \\ (ac)_{21} \\ (ac)_{22} \\ d \\ g \end{pmatrix} \quad \text{and} \quad e = \begin{pmatrix} e_{111} \\ e_{112} \\ e_{121} \\ e_{122} \\ e_{131} \\ e_{132} \\ e_{211} \\ e_{212} \\ e_{221} \\ e_{222} \\ e_{231} \\ e_{232} \end{pmatrix}.$$

Constants to be fitted have been arbitrarily divided into two sets. The B_1 set of constants includes only those constants that can be conveniently absorbed, μ or α and all other constants which represent the class or subclasses effects. The X matrix is therefore composed of only zeros and ones giving the class or subclass within which each of the observations fall. The B_2 set of constants includes all constants to be fitted other than those to be absorbed. The need for grouping all constants to be fitted into these two sets will become clear as the absorption process and completion of the least-squares analysis is presented in general matrix notation.

Least-Squares Equations

The least-squares equations for model (5) are given in tabular form below:

	$\hat{\alpha}$	$\hat{a}_i$	$\hat{b}_{ij}$	$\hat{c}_k$	$(\hat{ac})_{ik}$	$\hat{d}$	$\hat{g}$	RHM
α:	$n_{...}$	$n_{i..}$	$n_{ij.}$	$n_{..k}$	$n_{i.k}$	$D_{...}$	$G_{...}$	$Y_{...}$
a_i:		${}_0n_{i..}{}^0$	$n_{ij.}$	$n_{i.k}$	$n_{i.k}$	$D_{i..}$	$G_{i..}$	$Y_{i..}$
b_{ij}:			${}_0n_{ij.}{}^0$	n_{ijk}	n_{ijk}	$D_{ij.}$	$G_{ij.}$	$Y_{ij.}$
c_k:				${}_0n_{..k}{}^0$	$n_{i.k}$	$D_{..k}$	$G_{..k}$	$Y_{..k}$
$(ac)_{ik}$:					${}_0n_{i.k}{}^0$	$D_{i.k}$	$G_{i.k}$	$Y_{i.k}$
d:						$\sum_{ijkl}\sum\sum\sum D_{ijkl}^2$	$\sum_{ijkl}\sum\sum\sum D_{ijkl}G_{ijkl}$	$\sum_{ijkl}\sum\sum\sum D_{ijkl}y_{ijkl}$
g:							$\sum_{ijkl}\sum\sum\sum G_{ijkl}^2$	$\sum_{ijkl}\sum\sum\sum G_{ijkl}y_{ijkl}$

Elements to the left of the main diagonal in the coefficient matrix have been omitted since they would be the same as those given on the right of the main diagonal.

Before direct inversion of the least-squares coefficient matrix can be made or a unique solution of the equations can be obtained it is necessary to impose certain restrictions. An appropriate set of restrictions is that $\sum_i \hat{a}_i = \sum_j \hat{b}_{ij} = \sum_k \hat{c}_k = \sum_i (\hat{ac})_{ik} = \sum_k (\hat{ac})_{ik} = 0$. The subtractions and additions required among the coefficients and RHM's in the least-squares equations when these restrictions are imposed have previously been described except for the restriction that $\sum_j b_{ij} = 0$. When this restriction is imposed on the b_{ij} the coefficients for each of the $\hat{b}_{iq_i}$ are subtracted from the other coefficients for $\hat{b}_{ij}$, but only within the i^{th} A class of effects, i.e., the coefficients for the last $\hat{b}_{ij}$ in each A class are subtracted from the coefficients for the other $\hat{b}_{ij}$ within the corresponding A class. After this subtraction by columns the same type of subtraction is completed by rows, in which case the RHM's are also involved.

If the total number of equations for α, a_i and b_{ij} is large the inverse of the complete variance-covariance matrix and the estimates of α, a_i and b_{ij} can often be obtained most easily by an indirect but general method. The least-squares equations for the general model (6) in matrix notation are

$$\begin{bmatrix} D & N \\ N' & S \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

where, for the present example, D is a diagonal matrix of the $n_{ij.}$ of order s x s; N is the matrix of coefficients for $\hat{c}_k$, $(\hat{ac})_{ik}$, $\hat{d}$ and $\hat{g}$ in the $\alpha+a_i+b_{ij}$ equations of order s X (r+t+2); N' is the transpose of N (rows and columns reversed); S is the matrix of coefficients for $\hat{c}_k$, $(\hat{ac})_{ik}$, $\hat{d}$ and $\hat{g}$ in the c_k, $(ac)_{ik}$, d and g equations of order (r+t+2) X (r+t+2); Y_1 is the matrix of RHM's

for the $\alpha + a_i + b_{ij}$ equations, the $Y_{ij..}$ of order s X u with u being the number of dependent variables; Y_2 is the matrix of RHM's for the other equations of order (r+t+2) X u.

Computation of the Matrix Inverse and Estimates of Constants

Programs are generally available on high speed electronic computers for inverting matrices directly when there are no more than 30 or 40 equations. A few special programs are available, for some computers, which will invert matrices of considerably larger size. However, the cost of computing the inverse matrix directly when there are more than about 40 equations is usually very expensive. With large least-squares matrices it is often best to compute the inverse matrix by an indirect procedure.8/ Suppose the matrix inverse of the variance-covariance matrix under model (6) is partitioned as follows:

$$\begin{bmatrix} D & N \\ N' & S \end{bmatrix}^{-1} = \begin{bmatrix} A & G \\ G' & C \end{bmatrix}.$$

With this partitioning of the inverse it can be shown that

$$C = (S - N'D^{-1}N)^{-1}$$
$$G = - D^{-1}NC$$
$$A = D^{-1}\ (I - NG')$$

where I is the identity or unit matrix. It must be realized, of course, that restrictions must be imposed on the $S - N'D^{-1}N$ equations before the inverse C can be obtained. The changes made in the $S - N'D^{-1}N$ equations as a result of restrictions imposed, must also be made in N or the additional rows and columns must be added to C before computing G and A.

Although the indirect procedure described above for computing an inverse is general and can be applied to any set of symmetrical equations it is usually most useful for least-squares analyses when D is a diagonal matrix. When this is true, the procedure for computing the "new" coefficients in the remaining equations after the equations for a set of classes or subclasses are absorbed is given by $S - N'D^{-1}N$. With D as a diagonal matrix the calculation of $S - N'D^{-1}N$ can be completed most readily with the aid of simplified formulas. For example, under model (5) the absorption of the equations for $\alpha + a_i + b_{ij}$ is accomplished by computing the following "new" coefficients for $\hat{c}_k$, $(\hat{ac})_{ik}$, d and g:

$$C(c_k c_k) = n_{..k} - \sum_{ij}\sum \frac{n^2_{ijk}}{n_{ij.}}$$

$$C(c_k c_{k'}) = - \sum_{ij}\sum \frac{n_{ijk}n_{ijk'}}{n_{ij.}}$$

$$C\left[c_k(ac)_{ik}\right] = C\left[(ac)_{ik}(ac)_{ik}\right] = n_{i.k} - \sum_j \frac{n^2_{ijk}}{n_{ij.}}$$

8/ This indirect procedure was first given to the author by Dr. C. R. Henderson of Cornell University.

$$C[c_k(ac)_{ik'}] = C[(ac)_{ik}(ac)_{ik'}] = -\sum_j \frac{n_{ijk}n_{ijk'}}{n_{ij.}}$$

$$C(c_k d) = D_{..k} - \sum_{ij}\sum \frac{n_{ijk}D_{ij.}}{n_{ij.}}$$

$$C(c_k g) = G_{..k} - \sum_{ij}\sum \frac{n_{ijk}G_{ij.}}{n_{ij.}}$$

$$C[(ac)_{ik}(ac)_{i'k}] = C[(ac)_{ik}(ac)_{i'k'}] = 0$$

$$C[(ac)_{ik}d] = D_{i.k} - \sum_j \frac{n_{ijk}D_{ij.}}{n_{ij.}}$$

$$C[(ac)_{ik}g] = G_{i.k} - \sum_j \frac{n_{ijk}G_{ij.}}{n_{ij.}}$$

$$C(d\ d) = \sum_{ijkl}\sum\sum\sum D^2_{ijkl} - \sum_{ij}\sum \frac{D^2_{ij.}}{n_{ij.}}$$

$$C(d\ g) = \sum_{ijkl}\sum\sum\sum D_{ijkl}G_{ijkl} - \sum_{ij}\sum \frac{D_{ij.}G_{ij.}}{n_{ij.}}$$

$$C(g\ g) = \sum_{ijkl}\sum\sum\sum G^2_{ijkl} - \sum_{ij}\sum \frac{G^2_{ij.}}{n_{ij.}}$$

Under the general model (6) the "new" right hand members of the equations for the B_2 set of constants are obtained from $Y_2 - N'D^{-1}Y_1$; whereas, under the specific model (5) the "new" RHM's for the c_k, $(ac)_{ik}$, d and g equations are

$$S(c_k) = Y_{..k} - \sum_{ij}\sum \frac{n_{ijk}Y_{ij.}}{n_{ij.}}$$

$$S(ac)_{ik} = Y_{i.k} - \sum_j \frac{n_{ijk}Y_{ij.}}{n_{ij.}}$$

$$S(b) = \sum_{ijkl}\sum\sum\sum D_{ijkl}y_{ijkl} - \sum_{ij}\sum \frac{D_{ij.}Y_{ij.}}{n_{ij.}}$$

$$S(g) = \sum_{ijkl}\sum\sum\sum G_{ijkl}y_{ijkl} - \sum_{ij}\sum \frac{G_{ij.}Y_{ij.}}{n_{ij.}}$$

It should be noted that $n_{ij.}$ appears in the denominator in each of the $N'D^{-1}N$ and $N'D^{-1}Y_1$ terms given above for model (5) and that a product or square appears in the numerator in each case. This simplifies the computations

required in the absorption of a set of class or subclass effects.

When D is a diagonal matrix the matrix multiplications required to compute G and A are greatly simplified. Hence, the matrix inverse to the complete variance-covariance matrix can often be obtained with a minimum of time and labor even with a large number of least-squares equations.

Estimates of the constants included in the B_2 set under model (6) may be obtained from

$$\hat{B}_2 = (S - N'D^{-1}N)^{-1} \; (Y_2 - N'D^{-1}Y_1).$$

If G has been computed these estimates may also be obtained from $Y_1'G + Y_2'C$ or $G'Y_1 + CY_2$. When the estimates of constants included in B_2 are computed in both ways a check is provided on the computation of G.

Estimates of the constants included in the B_1 set may be obtained by solving the equations for the B_1 constants after estimates of the B_2 constants have been computed, i.e.,

$$\hat{B}_1 = D^{-1} \, (Y_1 - N\hat{B}_2).$$

If the G and A segments of the inverse have been computed these constants may also be obtained from $Y_1'A + Y_2'G'$ or $AY_1 + GY_2$. These different methods of computing $\hat{B}_1$ provide a means of checking the calculations involved in computing G and A.

In order to simplify the absorption process and the computation of A and G it is necessary, under model (5), to place the constants α, a_i and b_{ij} together. Hence, for this model, $\hat{B}_1$ consists of the least-squares subclass means, the $\hat{\alpha} + \hat{a}_i + \hat{b}_{ij}$. To obtain a separation of these constant estimates one can impose the restrictions that $\sum_i \hat{a}_i = \sum_j \hat{b}_{ij} = 0$ and compute averages of the $\hat{\alpha} + \hat{a}_i + \hat{b}_{ij}$, i.e.,

$$\hat{\alpha} + \hat{a}_i = \frac{\sum_j(\hat{\alpha}+\hat{a}_i+\hat{b}_{ij})}{q_i}$$

$$\hat{\alpha} = \frac{\sum_i\sum_j(\hat{\alpha}+\hat{a}_i+\hat{b}_{ij})}{\sum_i q_i} = \frac{\sum_i(\hat{\alpha} + \hat{a}_i)}{p} .$$

The estimates for each of the a_i and b_j can then be computed.

With model (5) the inverse matrix segments A and G apply to the $\hat{\alpha}+\hat{a}_i+\hat{b}_{ij}$ rather than to $\hat{\alpha}$, the $\hat{a}_i$ and $\hat{b}_{ij}$ separately. In many cases this may be all that is required. However, the transformation of these inverse segments to obtain the inverse elements which apply directly to α, the $\hat{a}_i$ and b_{ij} is easily accomplished. The general method of transforming sub-inverse matrices, such as A and G, was presented in previous sections and will not be repeated here for this special case.

Computation of Sums of Squares

When constants are fitted for all degrees of freedom among a set of subclasses (two factor, three factor, etc.) except the highest order interaction, the least-squares sum of squares for that interaction can be computed by a difference, as was shown for the two-way classification. Hazel (1946)[9] gave a method of computing the least-squares sum of squares for a two-factor interaction when constants were fitted for two main effects and a partial regression. The method of Hazel requires that the expected subclass means be computed from the constant estimates obtained when the highest order interaction is assumed to be non-existent. The sum of squares for interaction is then computed by multiplying the squared difference between the observed subclass mean and the expected subclass mean by the number of observations in the subclass and accumulating over all subclasses. This method of computing the sum of squares for the highest order interaction when constants for all other effects among the smallest subclasses have been fitted is easily extended to more complex analyses. However, it should be pointed out that no method is available for obtaining unbiased tests of significance for two or more interactions, unless constants are fitted for all interactions except one. Even then, if the test of significance for the interaction omitted (usually the highest order interaction) is significant and constants were not fitted simultaneously for this interaction the estimates of constants for all other effects are biased.

The appropriate error sum of squares, for use in testing the significance of the BC:A interaction omitted in the least-squares analysis under model (5), is computed from

$$\sum_{ijkl}\Sigma\Sigma\Sigma y^2_{ijkl} - \sum_{ijk}\Sigma\Sigma \frac{Y^2_{ijk}}{n_{ijk}} - \hat{d}'\left(\sum_{ijkl}\Sigma\Sigma\Sigma D_{ijkl}y_{ijkl} - \sum_{ijk}\Sigma\Sigma \frac{D_{ijk}Y_{ijk}}{n_{ijk}}\right)$$

$$- \hat{g}'\left(\sum_{ijkl}\Sigma\Sigma\Sigma G_{ijkl}y_{ijkl} - \sum_{ijk}\Sigma\Sigma \frac{G_{ijk}Y_{ijk}}{n_{ijk}}\right)$$

where $\hat{d}'$ and $\hat{g}'$ are the partial regressions of y on D and G computed on a within ABC subclass basis.

The sum of squares for BC:A interaction can most easily be computed directly from

$$R[\alpha,a_i,b_{ij},c_k,(ac)_{ik},(bc)_{ijk},d,g] - R[\alpha,a_i,b_{ij},c_k,(ac)_{ik},d,g]$$

where

$$R[\alpha,a_i,b_{ij},c_k,(ac)_{ik},(bc)_{ijk},d,g] = \sum_{ijk}\Sigma\Sigma \frac{Y^2_{ijk}}{n_{ijk}} + R(d',g'),$$

rather than by the indirect method of Hazel.

The error sum of squares when the BC:A interaction does not exist is computed from

$$\sum_{ijkl}\Sigma\Sigma\Sigma\, y^2_{ijkl} - R[\alpha,a_i,b_{ij},c_k,(ac)_{ik},d,g]$$

where the total reduction in sum of squares due to fitting all constants is computed by summing the products of the constant estimates and the corresponding RHM's in the usual manner. The mean square resulting from dividing the

9/ Hazel, L. N. The covariance analysis of multiple classification tables with unequal subclass numbers. Biometrics Bulletin 2:21-25. 1946.

sum of squares obtained in this way by the degrees of freedom for error is biased upwards if the BC:A interaction exists. Hence, the error mean square used by Hazel to test the significance of his two-factor interaction was inappropriate and would have been biased upwards if the interaction had existed.

The least-squares (or adjusted) sums of squares for A, B:A, C, AC, regression d and regression g may be computed by the general direct method, $B'Z^{-1}B$, previously described, when the inverse of the complete variance-covariance matrix is available. The adjusted sums of squares for those sources of variation which have only a few degrees of freedom can almost always be computed most easily by this direct method. In many cases, however, when model (5) is applicable there will be a considerable number of degrees of freedom for B:A as compared with each of the other sources of variation. In this case it may be somewhat easier to compute the sum of squares for B:A by the indirect procedure, i.e., from

$$R[\alpha, a_i, b_{ij}, c_k, (ac)_{ik}, d, g] - R[\alpha, a_i, c_k, (ac)_{ik}, d, g].$$

The decision concerning which procedure to use in each case is largely determined from the rank of the matrix inverse segment(s) that must be inverted if the direct method is used, as compared with the number of equations which must be solved if the indirect method is used. For example, if $p = 2$, $q_1 = 10$, $q_2 = 15$, and $r = 3$ the indirect method should definitely be used to compute the sum of squares for B:A, since it would require the solution of only 8 equations. The direct method would require the inversion of one 9x9 matrix and one 14x14 matrix as well as the matrix multiplications, $B'Z^{-1}B$, to obtain the same sum of squares.

Least-Squares Means, Standard Errors and Individual Comparisons

When all effects are fixed, except for the errors, the investigator is primarily concerned with the least-squares means with their standard errors and tests for the significance of contrasts among the means. The least-squares means are constructed from the constant estimates, e.g., under model (5)

$\hat{\mu} = \hat{\alpha} + \hat{d}\,\bar{D} + \hat{g}\,\bar{G}$ - - - - - - - - - - - Overall mean.

$\hat{\mu} + \hat{a}_i$ - - - - - - - - - - - - - - - Means of the A classes.

$\hat{\mu} + \hat{a}_i + \hat{b}_{ij}$ - - - - - - - - - - - - Means of the AB subclasses.

$\hat{\mu} + \hat{c}_k$ - - - - - - - - - - - - - - - Means of the C classes.

$\hat{\mu} + \hat{a}_i + \hat{c}_k + (\widehat{ac})_{ik}$ - - - - - - - - Means of the AC subclasses.

In each case, these means estimate the class or subclass arithmetic mean that wouldbe expected if a new set of data was taken in the same manner, except that equal subclass numbers would be achieved and the mean of the D_{ijkl} and G_{ijkl} for all classes and subclasses would equal $\bar{D}$ and $\bar{G}$. When unequal frequencies and variations in the means for D and G are a result of chance rather than a result of the effects being considered, these means (which are referred to here as the least-squares means) are the ones desired. Unequal weighting of the individual effects in obtaining expected means may sometimes be desirable. For example, if the a_i represent sire effects and the c_k represent age of dam

effects one could logically be interested in computing the expected sire means when each sire is mated to different numbers of dams of the different age classes. In this case,

$$\hat{\mu}' = \hat{\alpha} + \hat{d}\,\bar{D} + \hat{g}\,\bar{G} + \sum_k w_k \hat{c}_k$$

where the w_k are the weights to be given each age of dam effect based on the number for each age of dam class that is to be considered and $\sum_k w_k = 1$. The variance of $\hat{\mu}' + \hat{a}_i$ would be different from the variance of $\hat{\mu} + \hat{a}_i$, of course, but could still be computed from the inverse elements of the matrix inverse by making use of the general formula for computing the variance of a sum.

Standard errors for least-squares means, or for differences among the constant estimates within a set, are computed from the inverse elements and the estimate of $\hat{\sigma}_e^2$ in the usual manner that has previously been described. Duncan's Multiple Range Test, as modified by Kramer[10] may be used to make pairwise comparisons or tests of significance for orthogonal comparisons can be completed. Methods for completing these tests in least-squares analyses with fixed effects were also described in previous sections.

Estimation of Variance Components

When all effects in model (5) are random, except α, d and g the expectations of the mean squares, E(MS), obtained with a complete least-squares analysis are as follows:

	d.f.	E(MS)
A	p-1	$\sigma_e^2 + k_7\sigma_{ac}^2 + k_8\sigma_{b:a}^2 + k_9\sigma_a^2$
B:A	s-p	$\sigma_e^2 + k_6\sigma_{b:a}^2$
C	r-1	$\sigma_e^2 + k_4\sigma_{ac}^2 + k_5\sigma_c^2$
AC	t-p-r+1	$\sigma_e^2 + k_3\sigma_{ac}^2$
Regression d	1	$\sigma_e^2 + k_2\sigma_d^2$
Regression g	1	$\sigma_e^2 + k_1\sigma_g^2$
Error	$n_{...}$-s-t+p+3	σ_e^2

These expectations point out that exact tests of significance are available for each of the two regression coefficients, the AC interaction and the B:A effects. Since k_3, k_4, and k_7 are all different coefficients, with unequal subclass numbers as are also k_6 and k_8, no exact test of significance is available for the effects of A or C. If the interaction of A and C is non-existent the error

10/ Kramer, C. Y. 1957. Ibid.

mean square is the appropriate mean square for testing C effects. However, since each sum of squares in the complete least-squares analysis has been adjusted for possible effects of all other sources of variation considered this may give an inefficient test depending largely on the amount of unequal frequencies existing in the data. In fact, if any of the effects included in the model are really non-existent all tests of significance and estimates of variance components are less precisely determined than would be the case if such effects are omitted from the model and the data reanalyzed. A short-cut matrix multiplication procedure for use in deleting sets of effects is presented in the next major sub-section so that more precise results can be obtained.

Since the effects of the regressions are fixed there is no reason for computing k_1 and k_2. All other k's are needed in order to obtain the estimates of the variance components. The coefficients k_3, k_5, k_6 and k_9 may be obtained by the direct method, as previously described, at the same time the sums of squares are computed. However, if the number of B classes in the different A classes is large relative to the number of A and C classes it will be best to compute k_6 and the sum of squares for B:A by the indirect method. The remaining coefficients k_4, k_7 and k_8 can only be obtained by the indirect method at present if any of the $n_{i.k} = 0$. If all $n_{i.k} > 0$, $k_4 = \frac{1}{p} k_5$ and $k_7 = \frac{1}{r} k_9$. Even though all $n_{i.k} > 0$, k_8 must still be computed by the indirect method.

Mixed Model

In many problems the a_i and b_{ij} effects are random and the c_k effects, as well as the regressions, are fixed. For this case, Henderson11/ has presented a short-cut method (referred to by him as Method 2) of computing unbiased estimates of the variance components when p and the q_i are large. The method is also applicable when all effects are random.

Method 2 of Henderson and the complete least-squares analysis give unbiassed estimates but not "precisely the same estimates" of the variance components.

Using Method 2 of Henderson, the analysis of a set of data under model (5) when the a_i, the b_{ij} and e_{ijkl} are random and all other effects are fixed, may be completed with the following steps:

11/ Henderson, C. R. 1953. Ibid.

Step 1. Absorb the equations for $\alpha + a_i + b_{ij}$ into the equations for the c_k, the $(ac)_{ik}$, d and g in the manner described above. In general, matrix notation for any analysis this step is accomplished by computing the $S - N'D^{-1}N$ and the $Y_2 - N'D^{-1}Y_1$ values.

Step 2. Impose the restrictions (or constraints) on the constant estimates so that $\sum_k \hat{c}_k = \sum_i (\hat{ac})_{ik} = \sum_k (\hat{ac})_{ik} = 0$ and complete the appropriate subtractions and additions by columns and rows among the "new" coefficients and "new" right hand members.

Step 3. Invert the reduced coefficient matrix for the $c_k, (ac)_{ik}$, d and g and compute the estimates for these constants either from a direct solution of the equations or from $C^{-1}(Y_2 - N'D^{-1}Y_1)$; where C^{-1} is the matrix inverse and $(Y_2 - N'D^{-1}Y_1)$ refers to the right hand members.

Step 4. Compute the least-squares sums of squares for C, AC, Regression d, Regression g and Error. The error sum of squares is computed from

$$\text{Error S. Sqs.} = \sum_{ijkl}\sum\sum\sum y_{ijkl}^2 - R\left[\alpha, a_i, b_{ij}, c_k, (ac)_{ik}, d, g\right]$$

$$= \sum_{ijkl}\sum\sum\sum y_{ijkl}^2 - \sum_{ij}\sum \frac{Y_{ij.}^2}{n_{ij.}} - \hat{B}_2'(Y_2 - N'D^{-1}Y_1)$$

where $\hat{B}_2'$ is a row vector of the $\hat{c}_k, (\hat{ab})_{ik}$, $\hat{d}$ and $\hat{g}$ and $Y_2 - N'D^{-1}Y_1$ is a column vector of the corresponding RHM's in the reduced equations.

In most cases, the least-squares sums of squares for C, AC and the two regressions can most easily be computed by the general direct method, $B'Z^{-1}B$, as previously described.

Step 5. Set up the analysis of variance for the fixed effects and compute the F values for test of significance as indicated below:

Source of Variation	d.f.	E(MS)
C	r-1	$\sigma_e^2 + k_4\sigma_{ac}^2 + k_5\sigma_c^2$
AC	t-p-r+1	$\sigma_e^2 + k_3\sigma_{ac}^2$
Regression d	1	$\sigma_e^2 + k_2\sigma_d^2$
Regression g	1	$\sigma_e^2 + k_1\sigma_g^2$
Error	$n_{...}$-s-t+p+3	σ_e^2

If the interaction of A and C is likely to be non-significant and small, one should first compute the sums of squares for error and the AC interaction. When the F test clearly indicates that the $(ac)_{ik}$ effects are near zero, the $(ac)_{ik}$ effects can be deleted

before completing the test of significance for the fixed effects. A short-cut procedure for use in deleting a set of effects is given in the next sub-section.

No exact test of significance exists for the c_k fixed effects when the a_i effects are random and the $(ac)_{ik}$ do not equal zero. However, the $\hat{c}_k$ are unbiased estimates of the effects of the C classes and may therefore be used to adjust either these data or other data of a similar nature. If the AC interaction is found to be non-significant and small and is then deleted from the analysis, an exact test of significance can be made for the c_k effects. In this case, mean separation procedures for pairwise comparisons or tests of significance for orthogonal contrasts among the $\hat{c}_k$ may be completed too as has previously been outlined.

Step 6. Compute the adjusted totals $(Y''_{ij.})$ for the AB sub-classes as follows:

$$Y''_{ij.} = Y_{ij.} - \sum_k n_{ijk}\, \hat{c}_k - \sum_k n_{ijk}(\hat{ac})_{ik} - \hat{d}\, D_{ij.} - \hat{g}\, G_{ij.}$$

The general formula in matrix notation for computing the adjusted totals for the classes or subclasses absorbed is $Y_1'' = Y_1 - NB_2$.

The adjusted totals for the A classes are then computed by summing the $Y''_{ij.}$ over j, i.e.,

$$Y''_{i..} = \sum_j Y''_{ij.}$$

Step 7. Compute the "adjusted" sums of squares for A and B:A and the E(MS).

The adjusted sums of squares are computed as follows:

$$\text{Adjusted S. Sqs.} - - A' = \sum_i \frac{Y''^2_{i..}}{n_{i..}} - \frac{Y''^2_{...}}{n_{...}}$$

$$\text{Adjusted S. Sqs.} - - B':A' = \sum_{ij}\sum \frac{Y''^2_{ij.}}{n_{ij.}} - \sum_i \frac{Y''^2_{i..}}{n_{i..}} .$$

These adjusted sums of squares for A and B:A will not be the same as the least-squares sums of squares since this procedure, in effect, assumes that the adjustments for the effects of c_k, $(ac)_{ik}$, d and g are made without error. Since the $\hat{c}_k$, $(\hat{ac})_{ik}$, $\hat{d}$ and $\hat{g}$ are measured with error, the expectations of the mean squares for A' and B':A' are

	d.f.	E(MS)
A'	p-1	$k_{12}\sigma_e^2 + k_{13}\sigma^2_{b:a} + k_{14}\sigma_a^2$
B':A'	s-p	$k_{10}\sigma_e^2 + k_{11}\sigma^2_{b:a}$

The unusual feature of these expectations of mean squares is the coefficient for σ_e^2 in both the A' and the B':A' mean squares. Usually this coefficient is unity in each case, but since k_{10} and k_{12} are not unity it is necessary to compute them in order to obtain unbiased estimates of $\sigma^2_{b:a}$ and σ^2_a. In addition, it is necessary to compute k_{11}, k_{13}, and k_{14}. These last three coefficients are computed by the standard procedure for computing coefficients of variance components with unequal frequencies in the nested classification, i.e.,

$$k_{11} = \frac{1}{s-p}\left(n_{\cdot\cdot\cdot} - \sum_i \frac{\sum_j n_{ij\cdot}^2}{n_{i\cdot\cdot}}\right)$$

$$k_{13} = \frac{1}{p-1}\left(\sum_i \frac{\sum_j n_{ij\cdot}^2}{n_{i\cdot\cdot}} - \frac{\sum\sum_{ij} n_{ij\cdot}^2}{n_{\cdot\cdot\cdot}}\right)$$

$$k_{14} = \frac{1}{p-1}\left(n_{\cdot\cdot\cdot} - \frac{\sum_i n_{i\cdot\cdot}^2}{n_{\cdot\cdot\cdot}}\right).$$

The calculations required in the computation of k_{10} and k_{12} are somewhat more involved than in the computation of k_{11}, k_{13} and k_{14}. However, at this stage, many of the values required to compute k_{10} and k_{12} will have been computed during the absorption of the $\alpha + a_i + b_{ij}$ equations and the inversion of the reduced matrix, since

$$k_{10} = 1 + \frac{1}{s-p}\left[\sum\sum_{ij} C^{ij}(P''_{ij} - P'_{ij})\right]$$

$$k_{12} = 1 + \frac{1}{p-1}\left[\sum\sum_{ij} C^{ij}(P'_{ij} - P_{ij})\right].$$

The C^{ij} are the inverse elements in the matrix inverse to the reduced least squares coefficient matrix after absorption. The P''_{ij} elements are the values obtained in the absorption of the $\alpha+a_i+b_{ij}$ equations, i.e., the $N'D^{-1}N$ values; the P'_{ij} are the values obtained from the $N'D^{-1}N$ multiplication if the b_{ij} effects are deleted from the model and only the $\alpha + a_i$ equations were to be absorbed; and the P_{ij} are the values obtained from the $N'D^{-1}N$ multiplication if both the a_i and b_{ij} effects are deleted from the model and only the α

equation is absorbed. For example,

$$P_{11} = \frac{n_{..1}^2}{n_{...}} \qquad P''_{11} = \sum_i\sum_j \frac{n_{ij1}^2}{n_{ij.}}$$

$$P_{12} = \frac{n_{..1}n_{..2}}{n_{...}} \qquad P''_{12} = \sum_i\sum_j \frac{n_{ij1}n_{ij2}}{n_{ij.}}$$

etc.

$$P'_{11} = \sum_i \frac{n_{i.1}^2}{n_{i..}} \qquad \text{etc.}$$

$$P'_{12} = \sum_i \frac{n_{i.1}n_{i.2}}{n_{i..}}$$

etc.

The C and P matrices must all be of the same size. In this case, the order of these matrices may all be either (r+t+2) X (r+t+2) or (t-p+2) X (t-p+2), depending on whether the inverse elements for the columns and rows deleted as a result of imposing restrictions are computed and added back into C, or whether the same subtractions and additions are completed in the P matrices as was done in the $S-N'D^{-1}N$ matrix. Both procedures will yield the same results. In fact, these subtractions and additions by columns and rows in the P matrices may be completed in the $P''_{ij} - P'_{ij}$ and the $P'_{ij} - P_{ij}$ matrices, if desired.

Elimination or Deletion of a Set of Effects

If the test of significance for the AC interaction described above indicates that the $(ac)_{ik}$ effects probably equal zero the inverse elements associated with the $\hat{c}_k$, $\hat{d}$ and $\hat{g}$ may be adjusted by a deletion procedure. The adjusted inverse elements are the inverse elements that would be obtained if all equations for the $(ac)_{ik}$ are deleted in the reduced matrix and the matrix of coefficients for the c_k, d and g was inverted in the usual manner. The adjusted inverse elements are computed with the following matrix arithmetic:

$$C_A^{-1} = C_R^{-1} - RZ_{AC}^{-1} R'$$

where C_R^{-1} is the matrix of inverse elements associated with the $\hat{c}_k$, $\hat{d}$ and $\hat{g}$ in the C^{-1} matrix; Z_{AC}^{-1} is the inverse of the square segment of C^{-1} corresponding

to the $(ac)_{ik}$, and R is the matrix of inverse elements which associate the $(\hat{ac})_{ik}$ with the $\hat{c}_k$, $\hat{d}$ and $\hat{g}$.

Since Z^{-1}_{AC} is used to compute the least-squares sum of squares for AC, the additional labor required to obtain the new inverse matrix is often small indeed as compared with the direct inversion of the matrix of coefficients for the c_k, d and g. With other models the difference in time required to compute the reduced inverse matrix is likely to be considerably greater than with the particular model under discussion.

The estimates of the c_k, d and g unadjusted for the unequal frequencies associated with the AC interaction, may be obtained by multiplying the new inverse elements by the appropriate RHM's. Sums of squares for C, Regression d and Regression g are computed by the direct method from the $B'Z^{-1}B$ multiplication using the new constant estimates and the new inverse elements. Also, the sum of squares for error must be recomputed, since the degrees of freedom used for the AC interaction must now be placed into error.

Numerical Example

The average daily gain (ADG) for each of 65 steers of the Hereford breed arranged according to line, sire and age of dam is given in the table below. The age at weaning and initial weight at the beginning of the test feeding period is also given. All of these steers were fed for the same length of time in the feed lot. Identification numbers have been assigned for line, sire and steer for covenience. In addition, the original data have been rearranged, for illustrative purposes.

Line No.	Sire No.	Age of Dam	Steer No.	Age	Initial Weight	Average Daily Gain
1	1	3	1	192	390	2.24
			2	154	403	2.65
		4	3	185	432	2.41
			4	183	457	2.25
		5-up	5	186	483	2.58
			6	177	469	2.67
			7	177	428	2.71
			8	163	439	2.47
	2	4	9	188	439	2.29
			10	178	407	2.26
		5-up	11	198	498	1.97
			12	193	459	2.14
			13	186	459	2.44
			14	175	375	2.52
			15	171	382	1.72
			16	168	417	2.75
	3	3	17	154	389	2.38
		4	18	184	414	2.46
		5-up	19	174	483	2.29
			20	170	430	2.30

Line No.	Sire No.	Age of Dam	Steer No.	Age	Initial Weight	Average Daily Gain
			21	169	443	2.94
2	4	3	22	158	381	2.50
			23	158	365	2.44
		4	24	169	386	2.44
			25	144	339	2.15
		5-up	26	159	419	2.54
			27	152	469	2.74
			28	149	376	2.50
			29	149	375	2.54
	5	3	30	189	395	2.65
		4	31	187	447	2.52
			32	165	430	2.67
		5-up	33	181	453	2.79
			34	177	385	2.33
			35	151	414	2.67
			36	147	353	2.69
3	6	4	37	184	411	3.00
			38	184	420	2.49
		5-up	39	187	427	2.25
			40	184	409	2.49
			41	183	337	2.02
			42	177	352	2.31
	7	3	43	205	472	2.57
			44	193	340	2.37
		4	45	162	375	2.64
		5-up	46	206	451	2.37
			47	205	472	2.22
			48	187	402	1.90
			49	178	464	2.61
			50	175	414	2.13
	8	3	51	200	466	2.16
			52	184	356	2.33
			53	175	449	2.52
		4	54	178	360	2.45
		5-up	55	189	385	1.44
			56	184	431	1.72
			57	183	401	2.17
	9	3	58	166	404	2.68
		4	59	187	482	2.43
			60	186	350	2.36
			61	184	483	2.44
		5-up	62	180	425	2.66
			63	177	420	2.46
			64	175	449	2.52
			65	164	405	2.42

The mathematical model underlying the analysis to be made with these data is

$$y_{ijkl} = \alpha + g_i + s_{ij} + a_k + (ga)_{ik} + bA_{ijkl} + dW_{ijkl} + e_{ijkl}$$

$$i = 1, 2, 3 \qquad k = 1, 2, 3$$
$$j = 1, 2, 3, 4 \qquad l = 1, 2, \ldots, 6$$

where:

y_{ijkl} = the average daily gain for the l^{th} steer in the k^{th} age of dam class by the j^{th} sire of the i^{th} line,

α = the theoretical population mean with equal subclass frequencies when weaning age and initial weight both are equal to the absurd value of zero. The population mean with equal frequencies when the weaning age and initial weight both are equal to the average is

$$\mu = \alpha + b\bar{A} + d\bar{W},$$

g_i = effect of the i^{th} breeding group or line,

s_{ij} = effect of the j^{th} sire in the i^{th} line,

a_k = effect of the k^{th} age of dam class,

$(ga)_{ik}$ = interaction effects for line and age of dam,

b = partial regression of ADG on age at weaning. Since all steers were placed on test at the same time this regression coefficient will measure the effect of time of birth on ADG in the feed lot,

A_{ijkl} = age at weaning for a given steer,

d = partial regression of ADG on initial weight,

W_{ijkl} = initial weight for a given steer,

e_{ijkl} = random errors.

Complete Least-Squares Analysis

The least-squares equations are presented in Table 1. The reduced set of least-squares equations, resulting from imposing the restrictions that $\sum_i \hat{g}_i = \sum_j \hat{s}_{ij} = \sum_k \hat{a}_k = \sum_i (\widehat{ga})_{ik} = \sum_k (\widehat{ga})_{ik} = 0$ and after completing the appropriate subtractions and additions by columns and rows, is presented in Table 2. The matrix inverse to the reduced least-squares matrix is given in Table 3.

Table 1. Least-squares equations for the numerical example with the off-diagonal coefficients on the left omitted.

	$\hat{a}$	$\hat{g}_1$	$\hat{g}_2$	$\hat{g}_3$	$\hat{s}_{11}$	$\hat{s}_{12}$	$\hat{s}_{13}$	$\hat{s}_{21}$	$\hat{s}_{22}$	$\hat{s}_{31}$	$\hat{s}_{32}$	$\hat{s}_{33}$	$\hat{s}_{34}$	$\hat{a}_1$	$\hat{a}_2$	$\hat{a}_3$	$(\hat{ga})_{11}$	$(\hat{ga})_{12}$	$(\hat{ga})_{13}$	$(\hat{ga})_{21}$	$(\hat{ga})_{22}$	$(\hat{ga})_{23}$	$(\hat{ga})_{31}$	$(\hat{ga})_{32}$	$(\hat{ga})_{33}$	$\hat{b}$	$\hat{d}$	RHM
a:	65	21	15	29	8	8	5	8	7	6	8	7	8	12	16	37	3	5	13	3	4	8	6	7	16	11482	27096	156.74
g_1:		21			8	8	5							3	5	13	3	5	13							3725	9096	50.44
g_2:			15					8	7					3	4	8				3	4	8				2435	5987	38.17
g_3:				29						6	8	7	8	6	7	16							6	7	16	5322	12012	68.13
s_{11}:					8									2	2	4	2	2	4							1417	3501	19.98
s_{12}:						8									2	6		2	6							1457	3436	18.09
s_{13}:							5							1	1	3	1	1	3							851	2159	12.37
s_{21}:								8						2	2	4				2	2	4				1238	3110	19.85
s_{22}:									7					1	2	4				1	2	4				1197	2877	18.32
s_{31}:										6					2	4								2	4	1099	2356	14.56
s_{32}:											8			2	1	5							2	1	5	1511	3390	18.81
s_{33}:												7		3	1	3							3	1	3	1293	2848	14.79
s_{34}:													8	1	3	4							1	3	4	1419	3418	19.97
a_1:														12			3			3			6			2128	4810	29.49
a_2:															16			5			4			7		2848	6632	39.26
a_3:																37			13			8			16	6506	15653	87.99
$(ga)_{11}$:																	3									500	1182	7.27
$(ga)_{12}$:																		5								918	2149	11.67
$(ga)_{13}$:																			13							2307	5765	31.50
$(ga)_{21}$:																				3						505	1141	7.59
$(ga)_{22}$:																					4					665	1602	9.78
$(ga)_{23}$:																						8				1265	3244	20.80
$(ga)_{31}$:																							6			1123	2487	14.63
$(ga)_{32}$:																								7		1265	2881	17.81
$(ga)_{33}$:																									16	2934	6644	35.69
b :																										2,042,090	4,801,569	27,603.64
d :																											11,404,219	65,428.44

Table 2. Reduced set of least-squares equations.[a]

	$\hat{\alpha}$	$\hat{g}_1$	$\hat{g}_2$	$\hat{s}_{11}$	$\hat{s}_{12}$	$\hat{s}_{21}$	$\hat{s}_{31}$	$\hat{s}_{32}$	$\hat{s}_{33}$	$\hat{a}_1$	$\hat{a}_2$	$(\hat{ga})_{11}$	$(\hat{ga})_{12}$	$(\hat{ga})_{21}$	$(\hat{ga})_{22}$	$\hat{b}$	$\hat{d}$	RHM
α:	65	-8	-14	3	3	1	-2	0	-1	-25	-21	0	1	5	5	11482	27095	156.74
g_1:		50	29	3	3	0	2	0	1	0	1	-20	-17	-10	-9	-1597	-2916	-17.69
g_2:			44	0	0	1	2	0	1	5	5	-10	-9	-15	-13	-2887	-6025	-29.96
s_{11}:				13	5	0	0	0	0	0	0	0	0	0	0	566	1342	7.61
s_{12}:					13	0	0	0	0	-4	-2	-4	-2	0	0	606	1277	5.72
s_{21}:						15	0	0	0	1	0	0	0	1	0	41	233	1.53
s_{31}:							14	8	8	-1	-1	1	1	1	1	-320	-1062	-5.41
s_{32}:								16	8	0	-3	0	3	0	3	92	-28	-1.16
s_{33}:									15	3	-1	-3	1	-3	1	-126	-570	-5.18
a_1:										49	37	-6	-3	-11	-8	-4378	-10843	-58.50
a_2:											53	-3	-5	-8	-11	-3658	-9021	-48.73
$(ga)_{11}$:												38	29	22	16	4	-426	-3.17
$(ga)_{12}$:													41	16	23	280	147	-1.95
$(ga)_{21}$:														33	24	1051	2054	7.85
$(ga)_{22}$:															35	1069	2121	6.86
b:																2042090	4,801,569	27,603.64
d:																	11,404,219	65,428.44

a/ The off-diagonal coefficients to the left of the main diagonal are omitted.

Table 3. Matrix inverse to the complete variance-covariance matrix.a/

	$\hat{\alpha}$	$\hat{g}_1$	$\hat{g}_2$	$\hat{s}_{11}$	$\hat{s}_{12}$	$\hat{s}_{21}$	$\hat{s}_{31}$	$\hat{s}_{32}$	$\hat{s}_{33}$	$\hat{a}_1$	$\hat{a}_2$	$(\hat{ga})_{11}$	$(\hat{ga})_{12}$	$(\hat{ga})_{21}$	$(\hat{ga})_{22}$	$\hat{b}$	$\hat{d}$
c:	<u>1,949,913</u>	33,623	-248,412	48,591	77,403	-226,949	-27,619	140,568	-3,557	-16,443	53,316	-220,954	123,313	144,493	-25,562	-2,357,111	-196,950
g_1:		<u>46,831</u>	-29,401	-10,820	851	2,259	-7,138	1,719	962	2,433	-1,100	25,224	169	-16,576	276	14,217	-14,934
g_2:			<u>62,328</u>	4,317	-8,499	7,884	-5,275	-6,594	1,059	-425	-2,611	-4,514	-7,081	13,497	3,666	152,929	-3,411
s_{11}:				<u>98,039</u>	-12,738	-1,393	-2,595	1,447	-700	-11,138	2,884	-20,628	5,482	9,553	-2,365	-1,169	-12,855
s_{12}:					<u>109,726</u>	-6,673	6,086	2,535	1,972	18,833	-4,461	24,636	-7,723	-8,588	4,497	-109,574	28,272
s_{21}:						<u>79,911</u>	-1,629	-6,608	-858	-6,197	-313	15,414	-9,050	-18,227	6,627	141,698	-5,698
s_{31}:							<u>134,969</u>	-41,784	-52,016	16,417	-7,047	-10,057	7,936	-9,091	6,394	-54,874	31,205
s_{32}:								<u>104,560</u>	-28,235	-4,415	7,447	-2,592	-2,074	8,206	-6,539	-71,476	-4,008
s_{33}:									<u>120,044</u>	-11,368	8,035	12,891	-7,468	13,783	-8,181	-20,485	9,163
a_1:										<u>60,114</u>	-34,394	10,960	-4,070	9,913	-5,273	-41,405	24,368
a_2:											<u>45,124</u>	-7,414	2,962	-2,914	9,531	-31,895	-694
$(ga)_{11}$:												<u>132,501</u>	-79,153	-80,915	45,079	109,860	8,224
$(ga)_{12}$:													<u>93,828</u>	48,384	-56,554	-78,912	3,245
$(ga)_{21}$:														<u>126,387</u>	-74,643	-104,524	9,430
$(ga)_{22}$:															<u>98,938</u>	21,297	-2,714
b:																<u>19,093</u>	-2,396
d:																	<u>1,503</u>

a/ Elements to the left of the main diagonal are omitted. All elements, except those in the $\hat{b}$ and $\hat{d}$ columns, are multiplied by 10^{-6}. The elements in the $\hat{b}$ and $\hat{d}$ columns are multiplied by 10^{-8}.

Estimates of the constants obtained from the solution of the reduced set of equations are as follows:12/

$\hat{\alpha}$ =	3.06106	$\hat{a}_1$ =	.06946
$\hat{g}_1$ =	-.08035	$\hat{a}_2$ =	-.00611
$\hat{g}_2$ =	.03730	$(\hat{ga})_{11}$ =	-.11005
$\hat{s}_{11}$ =	.09051	$(\hat{ga})_{12}$ =	.01314
$\hat{s}_{12}$ =	-.10750	$(\hat{ga})_{21}$ =	.02522
$\hat{s}_{21}$ =	-.11917	$(\hat{ga})_{22}$ =	-.08066
$\hat{s}_{31}$ =	.16584	$\hat{b}$ =	-.008154
$\hat{s}_{32}$ =	.04098	$\hat{d}$ =	.001969
$\hat{s}_{33}$ =	-.26909		

Estimates for the overall mean, μ, and the constants omitted by the subtractions are computed as follows:

$$\hat{\mu} = \hat{\alpha} + \hat{b}\bar{A} + \hat{d}\bar{W}$$

$$= 3.06106 - (.008154)(176.65) + (.001969)(416.85)$$

$$= 2.44143$$

$$\hat{g}_3 = -(-.08035 + .03730) = .04305$$
$$\hat{s}_{13} = -(.09051 - .10750) = .01699$$
$$\hat{s}_{22} = -(-.11917) = .11917$$
$$\hat{s}_{34} = -(.16584 + .04098 - .26909) = .06227$$
$$\hat{a}_3 = -(.06946 - .00611) = -.06335$$
$$(\hat{ga})_{13} = -(-.11005 + .01314) = .09691$$
$$(\hat{ga})_{23} = -(.02522 - .08066) = .05544$$
$$(\hat{ga})_{31} = -(-.11005 + .02522) = .08483$$
$$(\hat{ga})_{32} = -(.01314 - .08066) = .06752$$
$$(\hat{ga})_{33} = -(.08483 + .06752) = -.15235$$

12/ Several extra decimal digits are given so that sums of squares computed later will be more accurate.

Analysis of Variance

A test of significance for the possible interaction of sires with age of dam (SA:G) will first be completed. The least-squares equations for b' and d' on a within line X sire X age of dam subclass basis are as follows:

$$4{,}830.38\ \hat{b}' + 8{,}046.82\ \hat{d}' = -19.5927$$

$$8{,}046.82\ \hat{b}' + 68{,}016.78\ \hat{d}' = 73.5537$$

The estimates of b' and d' are obtained by solving these two equations.

$$\hat{b}' = -.007295$$

$$\hat{d}' = .001945$$

The error sum of squares for testing the significance of the SA:G interaction is

$$\sum_{ijkl}\sum\sum\sum y^2_{ijk} - \sum_{ijk}\sum\sum \frac{Y^2_{ijk}}{n_{ijk}} - R(b',d')$$

$$= 2.2673 - (-.007295)(-19.5927) - (.001945)(73.5537)$$

$$= 2.2673 - .2860 = 1.9813$$

The GSA observed and expected subclass means are given in Table 4. The expected means are computed by first adjusting the subclass total (for Y) with the constant estimates. For example, the expected subclass mean for line 1, sire 1 and age of dam 3 is

$$\frac{1}{2}[4.89 + .08035 - .09051 - .06946 + .11005 + (.008154)(173.00 - 176.65)$$
$$- (.001969)(396.50 - 416.85)]$$
$$= 2.4654$$

The sum of squares for the SA:G interaction is computed from Table 4 as follows:

$$\text{S. Sqs.} - - - \text{SA:G} = (2)(-.0204)^2 + (2)(.0059)^2 + + + (4)(-.0183)^2$$
$$= .4890$$

This sum of squares may be more easily computed from 380.6325 + .2860 -380.4574 = .4611. Rounding errors probably account for the apparent descrepancy in the two methods.

The test of significance for the SA:G interaction may now be completed as follows:

Source of Variation	d.f.	S.Sqs.	M.S.	
SA:G	10	.4611	.04611	n.s.
Error	38	1.9813	.05214	

Table 4. Observed and expected GSA subclass means

Line No.	Sire No.	Age of Dam	n_{ijk}	Means Observed	Expected	Obs.-Exp.
1	1	3	2	2.4450	2.4654	-.0204
1	1	4	2	2.3300	2.3241	.0059
1	1	5	4	2.6075	2.5761	.0314
1	2	4	2	2.2750	2.3852	-.1102
1	2	5	6	2.2567	2.2846	-.0279
1	3	3	1	2.3800	2.3541	.0259
1	3	4	1	2.4600	2.5819	-.1219
1	3	5	3	2.5100	2.4815	.0285
2	4	3	2	2.4700	2.4307	.0393
2	4	4	2	2.2950	2.3507	-.0557
2	4	5	4	2.5800	2.5562	.0238
2	5	3	1	2.6500	2.5426	.1074
2	5	4	2	2.5950	2.5362	.0588
2	5	5	4	2.6200	2.5138	.1062
3	6	4	2	2.7450	2.6411	.1039
3	6	5	4	2.2675	2.2992	-.0317
3	7	3	2	2.4700	2.4526	.0174
3	7	4	1	2.6400	2.4575	.1825
3	7	5	5	2.2460	2.2851	-.0391
3	8	3	3	2.3367	2.3831	-.0464
3	8	4	1	2.4500	2.7376	-.2876
3	8	5	3	1.7767	1.9548	-.1781
3	9	3	1	2.6800	2.3589	.3211
3	9	4	3	2.4100	2.3648	.0452
3	9	5	4	2.5150	2.5333	-.0183

The ratio for F $\left(\frac{.04890}{.05214}\right)$ is less than one, indicating clearly that there is no interaction of sire by age of dam within lines. The analysis of variance can now be set up to test the significance of the effects for which constants have been fitted.

The sums of squares are computed from the sub-inverse matrices and the constant estimates by the general direct method, B'ZB, as follows:

$$\text{S.Sqs.--G} = \begin{bmatrix}-.08035 & .03730\end{bmatrix}\begin{bmatrix}.046831 & -.029401 \\ -.029401 & .062328\end{bmatrix}^{-1}\begin{bmatrix}-.08035 \\ .03730\end{bmatrix}$$

$$= \begin{bmatrix}-.08035 & .03730\end{bmatrix}\begin{bmatrix}30.33784 & 14.31079 \\ 14.31079 & 22.79476\end{bmatrix}\begin{bmatrix}-.08035 \\ .03730\end{bmatrix}$$

$$= \begin{bmatrix}-1.90385 & -.29963\end{bmatrix}\begin{bmatrix}-.08035 \\ .03730\end{bmatrix}$$

$$= .1418$$

$$\text{S.Sqs.--S:G} = \begin{bmatrix} .09051 & -.10750 \end{bmatrix} \begin{bmatrix} \underline{.098039} & -.041738 \\ -.041738 & \underline{.109726} \end{bmatrix}^{-1} B_1 + \frac{(-.11917)^2}{.079911}$$

$$+ \begin{bmatrix} .16584 & .04098 & -.26909 \end{bmatrix} \begin{bmatrix} \underline{.134969} & -.041784 & -.052016 \\ -.041784 & \underline{.104560} & -.028235 \\ -.052016 & -.028235 & \underline{.120044} \end{bmatrix}^{-1} B_3$$

$$= .13528 + .17772 + .62478$$

$$= .9378$$

$$\text{S.Sqs.--A} = \begin{bmatrix} .06946 & -.00611 \end{bmatrix} \begin{bmatrix} \underline{.06114} & -.034394 \\ -.034394 & \underline{.045124} \end{bmatrix}^{-1} B_A$$

$$= .1247$$

$$\text{S.Sqs.--GA} = \begin{bmatrix} -.11005 & .01314 & .02522 & -.08066 \end{bmatrix}$$

$$\begin{bmatrix} \underline{.132501} & -.079153 & -.080915 & .045079 \\ -.079153 & \underline{.093828} & .048384 & -.056554 \\ -.080915 & .048384 & \underline{.126387} & -.074643 \\ .045079 & -.056554 & -.074643 & \underline{.098938} \end{bmatrix}^{-1} B_{GA}$$

$$= .4651$$

$$\text{S.Sqs.--b} = \frac{(-.008154)^2}{.00019093} = .3482$$

$$\text{S.Sqs.--d} = \frac{(.001969)^2}{.00001503} = .2579$$

$$\text{S.Sqs.--Error} = \sum_i\sum_j\sum_k\sum_l y^2_{ijkl} - R(\text{all constants})$$

$$= 382.8998 - (3.06106)(156.74) - (-.08035)(-17.69) - - -$$

$$-(.001969)(65{,}428.44)$$

$$= 382.8998 - 380.3931$$

$$= 2.5067$$

The analysis of variance table may now be set up as follows:

Source of Variation	d.f.	S.Sqs.	M.S.	F
Lines (G)	2	.1418	.0709	<1
Sires (S:G)	6	.9530	.1588	2.99*
Ages of Dam (A)	2	.1247	.0624	1.20
GA	4	.4651	.1163	2.23
Regression b	1	.3482	.3482	6.67*
Regression d	1	.2579	.2579	4.94*
Error	48	2.5067	.0522	

The tests of significance assume that the effects of sires and error are random and that all other effects are fixed. The lack of significance for the GA interaction suggests that more accurate estimates of all other effects could be obtained by deleting the interaction. The short-cut deletion procedure will be illustrated in the next section with crossline data. The remaining analysis with this example will assume that the interaction effects, $(ga)_{ik}$, exist and must therefore be retained in the analysis.

Variance Component Estimates

If the classifications of G, S:G and A are regarded as random the investigator is interested in estimating σ^2_g, $\sigma^2_{s:g}$, σ^2_a and σ^2_{ga}. To compute estimates of these variance components from the complete least-squares analysis it is first necessary to compute the coefficients for them in the expectations of the mean squares. Since all of the subclasses of G and A are fitted and the number of degrees of freedom for G, S:G, A and GA are small it is fairly easy to compute estimates of $\sigma^2_{s:g}$, σ^2_a and σ^2_{ga}. The expectations of the mean squares for G, S:G, A, GA and errors are as follows:

	E(MS)
G	$\sigma^2_e + k_7\,\sigma^2_{ga} + k_8\,\sigma^2_{s:g} + k_9\,\sigma^2_g$
S:G	$\sigma^2_e + k_6\,\sigma^2_{s:g}$
A	$\sigma^2_e + k_4\,\sigma^2_{ga} + k_5\,\sigma^2_a$
GA	$\sigma^2_e + k_3\,\sigma^2_{ga}$
Error	σ^2_e

The coefficients k_3, k_5, k_6, and k_9 may easily be computed by the direct method for this particular analysis. For example,

$$k_9 = \frac{1}{3}\left[30.33784 + 22.79476 - \frac{1}{2}(14.31079 + 14.31079)\right]$$

$$= \frac{1}{3}\ (38.82181) = 12.941.$$

The values for k_3, k_5 and k_6, when computed in a similar manner from the inverse matrices used to compute the sum of squares directly, are

$$k_3 = 6.967$$
$$k_5 = 15.438$$
$$k_6 = 6.284$$

Since all GA subclasses are filled, the coefficients for σ^2_{ga}, k_4 and k_7, are computed as follows:

$$k_4 = \frac{1}{3} k_5 = 5.479$$

$$k_7 = \frac{1}{3} k_9 = 4.314$$

The only method available at present to compute k_8 is the indirect method of Henderson, which requires the inverse of the 15x15 least-squares coefficient matrix when the g_i are omitted from the model as well as the associated sums matrix of the same order. Since the indirect method was illustrated with the numerical example of the last section, k_8 will not be computed for this analysis. The estimates of σ^2_{ga}, σ^2_a and $\sigma^2_{s:g}$ are computed as follows:

	d.f.	Mean Squares	E(MS)
S:G	6	.1588	$\sigma^2_e + 6.342\ \sigma^2_{s:g}$
A	2	.0623	$\sigma^2_e + 5.479\ \sigma^2_{ga} + 15.438\ \sigma^2_a$
GA	4	.1163	$\sigma^2_e + 6.967\ \sigma^2_{ga}$
Error	48	.0509	σ^2_e

$\sigma^2_e = .0522$ $\quad\quad$ $\sigma^2_a = -.0026$

$\sigma^2_{ga} = .0092$ $\quad\quad$ $\sigma^2_{s:g} = .0170$

Method 2 of Henderson

Frequently with animal breeding data such as these, the g_i and s_{ij} will be genetic classifications for which variance component estimates are desired. All other effects in the model, except the errors, will be fixed. When the numbers of degrees of freedom for the genetic classifications are adequate the complete least-squares analysis will usually be impractical. The modified least-squares analysis developed by Henderson, which provides exact least-squares tests of significance for fixed effects when no interactions exist

among fixed and random effects and unbiased estimates of the variance components, will now be illustrated with the present set of data.

<u>Step 1.</u> Absorption of the $\alpha+g_i+s_{ij}$ equations.

The $N'D^{-1}N$ values are presented in Table 5 and the $S-N'D^{-1}N$ values (the reduced matrix) are presented in Table 6. These values were computed from the original least-squares equations in the same manner as was illustrated with the numerical example of the last section.

<u>Step 2.</u> Imposing the restrictions that $\sum_k \hat{a}_k = \sum_i (\widehat{ga})_{ik} = \sum_k (\widehat{ga})_{ik} = 0$.

The reduced set of equations for the a_k, $(ga)_{ik}$, b and d after completing the appropriate subtractions by columns and rows is presented in Table 7.

<u>Step 3.</u> Inversion of reduced matrix and calculation of constant estimates.

The inverse of the reduced matrix for the a_k, $(ga)_{ik}$, b and d, (C), is the same as given for this segment in Table 3. The estimates of these constants, obtained from solving the reduced set of equations, are identical with those obtained when the complete matrix was solved.

<u>Step 4.</u> Least-squares sums of squares.

The sums of squares for A, GA, Regression b and Regression d are the same as given in the complete least-squares analysis.

<u>Step 5.</u> Analysis of variance for fixed effects.

The analysis of variance for these effects is the same as was given for A, GA, Regression b, Regression d and error in the complete least-squares analysis. When the complete least-squares analysis is not done the error sum of squares is computed from

$$\sum_{ijk}\sum\sum y^2_{ijk} - \sum_{ij}\sum \frac{Y^2_{ij}}{n_{ij.}} - \hat{B}'_2 \, (Y_2 - N'D^{-1}Y_1)$$

as was indicated previously.

Since the g_i and b_{ij} are now assumed to be random no exact test of significance is available for the a_k effects.

<u>Step 6.</u> Computation of the adjusted totals for the GA subclasses and the G classes.

The adjusted totals for the line X sire subclasses are computed as follows:

$$\begin{aligned} Y''_{11.} = {} & 19.98 - (2)(.06946) - (2)(-.00611) - (4)(-.06335) \\ & -(2)(-.11005) - (2)(.01314) - (4)(.09691) - (-.008154)(1417) \end{aligned}$$

Table 5. Intermediate values ($N'D^{-1}N$) used in the absorption of the $\alpha+g_i+s_{ij}$ equations.

	$\hat{a}_1$	$\hat{a}_2$	$\hat{a}_3$	$(\hat{ga})_{11}$	$(\hat{ga})_{12}$	$(\hat{ga})_{13}$	$(\hat{ga})_{21}$	$(\hat{ga})_{22}$	$(\hat{ga})_{23}$	$(\hat{ga})_{31}$	$(\hat{ga})_{32}$	$(\hat{ga})_{33}$	$\hat{b}$	$\hat{d}$	RHM
a_1:	3.2536	2.5393	6.2071	.7000	.7000	1.6000	.6429	.7857	1.5714	1.9107	1.0536	3.0357	2,114.22	4,990.87	28.5860
a_2:		4.3310	9.1298	.7000	1.2000	3.1000	.7857	1.0714	2.1429	1.0536	2.0595	3.8869	2,812.25	6,663.24	38.9945
a_3:			21.6631	1.6000	3.1000	8.3000	1.5714	2.1429	4.2857	3.0357	3.8869	9.0774	6,555.53	15,440.89	89.1596
$(ga)_{11}$:				.7000	.7000	1.6000	0	0	0	0	0	0	524.45	1,307.05	7.4690
$(ga)_{12}$:					1.2000	3.1000	0	0	0	0	0	0	888.70	2,166.05	11.9915
$(ga)_{13}$:						8.3000	0	0	0	0	0	0	2,311.85	5,622.90	30.9795
$(ga)_{21}$:							.6429	.7857	1.5714	0	0	0	480.50	1,188.50	7.5796
$(ga)_{22}$:								1.0714	2.1429	0	0	0	651.50	1,599.50	10.1968
$(ga)_{23}$:									4.2857	0	0	0	1,303.00	3,199.00	20.3936
$(ga)_{31}$:										1.9107	1.0536	3.0357	1,109.27	2,495.32	13.5373
$(ga)_{32}$:											2.0595	3.8869	1,272.05	2,897.69	16.8062
$(ga)_{33}$:												9.0774	2,940.68	6,618.99	37.7865
b :													2,034,670.94	4,790,758.38	27,637.23
d :														11,312,307.63	65,338.54

Table 6. Least-squares equations ($S-N'D^{-1}N$) for the a_i, $(ga)_{ik}$, b and d after absorption of the $a+g_i+s_{ij}$ equations.

	$\hat{a}_1$	$\hat{a}_2$	$\hat{a}_3$	$(\hat{ga})_{11}$	$(\hat{ga})_{12}$	$(\hat{ga})_{13}$	$(\hat{ga})_{21}$	$(\hat{ga})_{22}$	$(\hat{ga})_{23}$	$(\hat{ga})_{31}$	$(\hat{ga})_{32}$	$(\hat{ga})_{33}$	$\hat{b}$	$\hat{d}$	RHM
a_1:	8.7464	-2.5393	-6.2071	2.3000	-.7000	-1.6000	2.3571	-.7857	-1.5714	4.0893	-1.0536	-3.0357	13.73	-180.87	.9040
a_2:		11.6690	-9.1298	-.7000	3.8000	-3.1000	-.7857	2.9286	-2.1429	-1.0536	4.9405	-3.8869	35.75	-31.24	.0655
a_3:			15.3369	-1.6000	-3.1000	4.7000	-1.5714	-2.1429	3.7143	-3.0357	-3.8869	6.9226	-49.53	212.11	-1.1690
$(ga)_{11}$:				2.3000	-.7000	-1.6000	0	0	0	0	0	0	-24.45	-125.05	-.1990
$(ga)_{12}$:					3.8000	-3.1000	0	0	0	0	0	0	29.30	-17.05	-.3215
$(ga)_{13}$:						4.7000	0	0	0	0	0	0	-4.85	142.10	.5205
$(ga)_{21}$:							2.3571	-.7857	-1.5714	0	0	0	24.50	-47.50	.0104
$(ga)_{22}$:								2.9286	-2.1429	0	0	0	13.50	2.50	-.4168
$(ga)_{23}$:									3.7143	0	0	0	-38.00	45.00	.4064
$(ga)_{31}$:										4.0893	-1.0536	-3.0357	13.73	-8.32	1.0927
$(ga)_{32}$:											4.9405	-3.8869	-7.05	-16.69	1.0038
$(ga)_{33}$:												6.9226	-6.68	25.01	-2.0965
b :													7,419.06	10,810.62	-33.5887
d :														91,911.37	89.9002

Table 7. Reduced equations for the a_i, $(ga)_{ik}$, b and d constants.a/

	$\hat{a}_1$	$\hat{a}_2$	$(\hat{ga})_{11}$	$(\hat{ga})_{12}$	$(\hat{ga})_{21}$	$(\hat{ga})_{22}$	$\hat{b}$	$\hat{d}$	RHM
a_1:	36.497620	28.134524	-6.883333	-4.091666	-7.869046	-6.148808	63.3166	-392.9833	2.073595
a_2:		45.265477	-4.091666	-4.936903	-6.148808	-8.708332	85.2869	-243.3524	1.435083
$(ga)_{11}$:			27.283333	21.491666	17.083333	12.791666	-40.0166	-233.8167	-3.908667
$(ga)_{12}$:				34.336903	12.791666	19.636903	34.5131	-117.4476	-3.942298
$(ga)_{21}$:					26.297620	19.434524	42.0834	-59.1667	-3.585239
$(ga)_{22}$:						30.565474	51.8631	-.7976	-3.923512
b :							7,419.06	10,810.62	-33.5887
d :								91,911.37	89.9022

a/ Two additional decimals were carried in the $S-N'D^{-1}N$ and $Y_2-N'D^{-1}Y_1$ calculations than are given in Tables 5 and 6.

$-(.001969)(3501)$

$= 24.5736$

$$Y''_{12.} = 18.09 - (2)(-.00611)-(6)(-.06335)-(2)(.01314)-(6)(.09691)$$

$$-(-.008154)(1457)-(.001969)(3436)$$

$= 22.9895$

$Y''_{13.} = 14.9909$ $\quad Y''_{31.} = 19.6223$ $\quad Y''_{34} = 25.3348$

$Y''_{21.} = 23.8369$ $\quad Y''_{32.} = 25.1643$

$Y''_{22} = 22.5260$ $\quad Y''_{33.} = 19.9128$

The adjusted totals for lines are

$Y''_{1..} = 24.5736 + 22.9895 + 14.9909 = 62.5540$

$Y''_{2..} = 46.3629$

$Y''_{3..} = 90.0342$

Step 7. Computation of sums of squares and variance component estimates.

The adjusted sums of squares for line (G') and sires within lines (S':G') may now be computed as follows:

$$\text{S. Sqs.--G'} = \frac{(62.5540)^2}{21} + \frac{(46.3629)^2}{15} + \frac{(90.0342)^2}{29} - \frac{(198.9511)^2}{65}$$

$$= 609.1574 - 608.9468$$

$$= .2106$$

$$\text{S. Sqs.--S':G'} = \frac{(24.5736)^2}{8} + \frac{(22.9895)^2}{8} + + + \frac{(25.3348)^2}{8} - 609.1574$$

$$= 610.2110 - 609.1574$$

$$= 1.0536$$

The analysis of variance for lines (G') and sires within lines (S':G') is set up as follows:

	d.f.	S.Sqs.	M.S.	E(MS)
G'	2	.2051	.1026	$k_{12}\sigma_e^2 + k_{13}\sigma_{s:g}^2 + k_{14}\sigma_g^2$
S':G	6	1.0876	.1813	$k_{10}\sigma_e^2 + k_{11}\sigma_{s:g}^2$
Error	48		.0522	σ_e^2

The coefficients for $\sigma_{s:g}^2$ and σ_g^2 om the E(MS) are computed as follows:

$$k_{11} = \frac{1}{6}\left(65 - \frac{(8)^2+(8)^2+(5)^2}{21} - \frac{(8)^2+(7)^2}{15} - \frac{(6)^2+(8)^2+(7)^2+(8)^2}{29}\right)$$

$$= \frac{1}{6}(65 - 22.1639)$$

$$= \frac{1}{6}(42.8361)$$

$$= 7.139$$

$$k_{13} = \frac{1}{2}\left(22.1639 - \frac{(8)^2+(8)^2+(5)^2 + + + (8)^2}{65}\right)$$

$$= \frac{1}{2}(22.1639 - 7.3692)$$

$$= \frac{1}{2}(14.7947)$$

$$= 7.397$$

$$k_{14} = \frac{1}{2}\left(65 - \frac{(21)^2+(15)^2+(29)^2}{65}\right)$$

$$= \frac{1}{2}(65 - 23.3185)$$

$$= \frac{1}{2}(41.6815)$$

$$= 20.841$$

The coefficients for σ_e^2, k_{10} and k_{12}, are computed from the segment of the matrix inverse to the variance-covariance matrix corresponding to the effects which remained after absorption and the P_{ij} matrices, i.e.,

$$k_{10} = 1 + \frac{1}{6}\left[\sum_{i}\sum_{j} C^{ij}(P''_{ij} - P'_{ij})\right]$$

and $$k_{12} = 1 + \frac{1}{2}\left[\sum_{i}\sum_{j} C^{ij}(P'_{ij} - P_{ij})\right].$$

The P'' matrix was computed during the absorption of the $\alpha+g_i+s_{ij}$ equations and is given in Table 5. The P' matrix is given in Table 8 and the P matrix is given in Table 9. These matrices were computed in the manner previously described from certain coefficients in the original set of least-squares equations (Table 1). The P''-P' matrix, after completing the appropriate subtractions by rows and columns, is given in Table 10. The reduced P'-P matrix is given in Table 11.

The coefficients, k_{10} and k_{12}, are now computed as follows:

$$k_{10} = 1 + \frac{1}{6}\Big[(.060114)(2.6256)+(-.034394)(.6191)+(.010960)(-.4303)$$

$$+ + + (.00001503)(7376)\Big]$$

$$= 1 + \frac{1}{6}\,(.7188)$$

$$= 1.120$$

$$k_{12} = 1 + \frac{1}{2}\Big[(.060114)(.2616)+(-.034394)(.1696)+(.010960)(1.3137)$$

$$+ + + (.00001503)(10{,}485)\Big]$$

$$= 1 + \frac{1}{2}\,(1.1379)$$

$$= 1.569$$

The computed mean squares may now be set equal to their expectations and the variance component estimates computed.

	d.f.	Mean Squares
G'	2	$.1026 = 1.603\,\hat{\sigma}_e^2 + 7.397\,\hat{\sigma}_{s:g}^2 + 20.841\,\hat{\sigma}_g^2$
S':G'	6	$.1813 = 1.136\,\hat{\sigma}_e^2 + 7.139\,\hat{\sigma}_{s:g}^2$
Error	48	$.0509 = \hat{\sigma}_e^2$

$$\hat{\sigma}_{s:g}^2 = \frac{.1813 - (1.136)(.0509)}{7.139} = .0173$$

$$\hat{\sigma}_g^2 = \frac{.1026 - (1.603)(.0509) - (7.397)(.0173)}{20.841} = -.0051$$

The estimate of .0173 for $\sigma_{s:g}^2$ is essentially the same as that obtained from the complete least-squares analysis where $\sigma_{s:g}^2$ was estimated to be .0166. The difference is due to different methods of estimation.

Table 8. The P' matrix resulting from the multiplication $N'D^{-1}N$ if the $\alpha + g_i$ equations were to be absorbed.

	$\hat{a}_1$	$\hat{a}_2$	$\hat{a}_3$	$(\hat{ga})_{11}$	$(\hat{ga})_{12}$	$(\hat{ga})_{13}$	$(\hat{ga})_{21}$	$(\hat{ga})_{22}$	$(\hat{ga})_{23}$	$(\hat{ga})_{31}$	$(\hat{ga})_{32}$	$(\hat{ga})_{33}$	$\hat{b}$	$\hat{d}$
a_1:	2.2700	2.9626	6.7675	.4286	.7143	1.8571	.6000	.8000	1.6000	1.2414	1.4483	3.3103	2,120.25	4,982.07
a_2:		3.9468	9.0906	.7143	1.1905	3.0952	.8000	1.0667	2.1333	1.4483	1.6897	3.8621	2,820.86	6,661.70
a_3:			21.1419	1.8571	3.0952	8.0476	1.6000	2.1333	4.2667	3.3103	3.8621	8.8276	6,540.89	15,451.23
$(ga)_{11}$:				.4286	.7143	1.8571	0	0	0	0	0	0	532.14	1,299.43
$(ga)_{12}$:					1.1905	3.0952	0	0	0	0	0	0	886.90	2,165.71
$(ga)_{13}$:						8.0476	0	0	0	0	0	0	2,305.95	5,630.86
$(ga)_{21}$:							.6000	.8000	1.6000	0	0	0	487.00	1,197.40
$(ga)_{22}$:								1.0667	2.1333	0	0	0	649.33	1,596.53
$(ga)_{23}$:									4.2667	0	0	0	1,298.67	3,193.07
$(ga)_{31}$:										1.2414	1.4483	3.3103	1,101.10	2,485.24
$(ga)_{32}$:											1.6897	3.8621	1,284.62	2,899.45
$(ga)_{33}$:												8.8276	2,936.28	6,627.31
b :													2,032,704	4,789,756
d :														11,304,932

Table 9. The P matrix resulting from the multiplication $N'D^{-1}N$ if only the α equation were to be absorbed.

	$\hat{a}_1$	$\hat{a}_2$	$\hat{a}_3$	$(\hat{ga})_{11}$	$(\hat{ga})_{12}$	$(\hat{ga})_{13}$	$(\hat{ga})_{21}$	$(\hat{ga})_{22}$	$(\hat{ga})_{23}$	$(\hat{ga})_{31}$	$(\hat{ga})_{32}$	$(\hat{ga})_{33}$	$\hat{b}$	$\hat{d}$
a_1:	2.2154	2.9538	6.8308	.5538	.9231	2.4000	.5538	.7385	1.4769	1.1077	1.2923	2.9538	2,119.75	5,002.15
a_2:		3.9385	9.1077	.7385	1.2308	3.2000	.7385	.9846	1.9692	1.4769	1.7231	3.9385	2,826.34	6,669.54
a_3:			21.0615	1.7077	2.8462	7.4000	1.7077	2.2769	4.5538	3.4154	3.9846	9.1077	6,535.91	15,423.31
$(ga)_{11}$:				.1385	.2308	.6000	.1385	.1846	.3692	.2769	.3231	.7385	529.94	1,250.54
$(ga)_{12}$:					.3846	1.0000	.2308	.3077	.6154	.4615	.5385	1.2308	883.23	2,084.23
$(ga)_{13}$:						2.6000	.6000	.8000	1.6000	1.2000	1.4000	3.2000	2,296.40	5,419.00
$(ga)_{21}$:							.1385	.1846	.3692	.2769	.3231	.7385	529.94	1,250.54
$(ga)_{22}$:								.2462	.4923	.3692	.4308	.9846	706.58	1,667.38
$(ga)_{23}$:									.9846	.7385	.8615	1.9692	1,413.17	3,334.77
$(ga)_{31}$:										.5538	.6462	1.4769	1,059.88	2,501.08
$(ga)_{32}$:											.7538	1.7231	1,236.52	2,917.92
$(ga)_{33}$:												3.9385	2,826.34	6,669.54
b :													2,028,251	4,786,228
d :														11,294,447

Table 10. The reduced P''-P' matrix.

	$\hat{a}_1$	$\hat{a}_2$	$(\hat{ga})_{11}$	$(\hat{ga})_{12}$	$(\hat{ga})_{21}$	$(\hat{ga})_{22}$	$\hat{b}$	$\hat{d}$
a_1:	2.6256	.6191	-.4303	.3855	-1.3492	-.0812	-20.67	19.14
a_2:		.8270	.3855	-.3177	-.0812	-.5655	-23.25	11.88
$(ga)_{11}$:			2.5063	.5953	1.4683	.1049	-17.36	-2.82
$(ga)_{12}$:				.8223	.1049	.5700	12.87	1.74
$(ga)_{21}$:					1.5874	.1286	-14.60	-33.23
$(ga)_{22}$:						.5745	14.81	-9.52
b:							1967	1002
d:								7376

Table 11. The reduced P'-P matrix.

	$\hat{a}_1$	$\hat{a}_2$	$(\hat{ga})_{11}$	$(\hat{ga})_{12}$	$(\hat{ga})_{21}$	$(\hat{ga})_{22}$	$\hat{b}$	$\hat{d}$
a_1:	.2616	.1696	1.3137	1.0908	.1415	.1530	-4.48	-48.00
a_2:		.1229	.7060	.5717	-.1548	-.1109	-10.46	-35.76
$(ga)_{11}$:			8.2102	6.9129	3.4481	3.1033	61.37	-189.36
$(ga)_{12}$:				5.8255	3.0264	2.7163	55.96	-154.14
$(ga)_{21}$:					4.7301	4.0521	140.28	62.17
$(ga)_{22}$:						3.4754	119.09	47.09
b:							4453	3528
d:								10,485

MODELS FOR THE ANALYSIS OF DATA FROM CROSSBREEDING EXPERIMENTS

The general least-squares procedures for the analysis of data from single crosses only were developed by Henderson.[13] The designs of many crossbreeding or line crossing experiments result in data which may be analyzed in several ways. Three methods of analysis will be considered in this section: first, the appropriate analysis for data obtained when all possible matings are made between lines but no linebreds are produced; second, an analysis of all data from linebreds and linecrosses produced together wherein the linebred effects are estimated independently of the general combining ability and maternal ability effects; and third, an analysis of all data from linebreds and linecrosses produced together wherein the linebred effects are estimated simultaneously with the general combining ability and maternal ability effects.

Analysis of Linecross Data

The analysis of data in which linebreds are not included will be considered first.

Model

The underlying mathematical model for the analysis when linebreds are not included is:

$$y_{ijk} = \mu + g_i + g_j + m_j + c_{ij} + r_{ij} + e_{ijk} \quad . \ . \ . \ . \ (7)$$

where y_{ijk} is the k^{th} observation on the progeny from the i^{th} line of sire and the j^{th} line of dam, μ is the over-all mean when equal subclass numbers exist, $g_i(g_j)$ is the general combining ability effect for the i^{th} (j^{th}) line, m_j is the maternal effect for the j^{th} line of dam, c_{ij} is the specific combining ability effect, r_{ij} is the sex-linkage or reciprocal effect and e_{ijk} is the random error, assumed to be NID$(0,\sigma_e^2)$. The $g_i(g_j)$ are assumed to be one-half the additive genetic value (breeding value) of the i^{th}(j^{th}) line, and the value is expressed as a deviation from μ. The m_j measure the pre-natal and post-natal mothering ability of a line, and each is a function of the genotype of a line rather than of the genes transmitted to the female progeny of the line.

The c_{ij} are effects in addition to the g_i and m_j effects, and are measured over the progeny of the i^{th} line of sire with the j^{th} line of dam and the progeny of the j^{th} line of sire with the i^{th} line of dam. The c_{ij} measure how much better or poorer than average are the means of reciprocal matings than would be expected on the basis of exact knowledge of the additive genetic values and maternal values of the lines. The r_{ij} are effects in addition to the additive genetic, maternal, and specific effects and are measures of the differences between reciprocal crosses after account has been taken of the

13/ Henderson, C. R. 1949. Ibid.

differences in maternal ability between the i^{th} line and the j^{th} line. For a more detailed discussion of these effects, see Henderson's thesis.

Least-Squares Equations

The least-squares equations for model (7) are presented in tabular form below:

	$\hat{\mu}$	$\hat{g}_i$	$\hat{m}_j$	$\hat{c}_{ij}$	$\hat{r}_{ij}$	RHM
μ	$n_{..}$	$n_{i.}+n_{.i}$ $n_{ij}+n_{ji}$	$n_{.j}$	$n_{ij}+n_{ji}$	n_{ij}	$Y_{..}$
g_i	$n_{i.}+n_{.i}$	$n_{i.}+n_{.i}$ $n_{ij}+n_{ji}$	n_{ij}	$n_{ij}+n_{ji}$	n_{ij}	$Y_{i.}+Y_{.i}$
m_j	$n_{.j}$	n_{ij}	${}_0n_{ij}{}^0$	n_{ij}	n_{ij}	$Y_{.j}$
c_{ij}	$n_{ij}+n_{ji}$	$n_{ij}+n_{ji}$	n_{ij}	${}_0n_{ij}+n_{ji}{}^0$	n_{ij}	$Y_{ij}+Y_{ji}$
r_{ij}	n_{ij}	n_{ij}	n_{ij}	n_{ij}	${}_0n_{ij}{}^0$	Y_{ij}

These equations differ somewhat from the least-squares equations presented previously, in that the off-diagonal elements in the segment of the coefficient matrix concerned with general combining ability effects are not zero. This is a result of the fact that each crossbred group is used in estimating the general combining ability effect of two lines. For example, a cross between a sire of line 1 and a dam of line 2 is used in estimating the general combining ability of both lines 1 and 2. Thus, the sum of the coefficients for the $\hat{g}_i$ in the μ equation is equal to twice the coefficient for $\hat{\mu}$. However, the sums for the coefficients for the $\hat{m}_j$, $\hat{c}_{ij}$, and $\hat{r}_{ij}$ effects in the μ equation are equal to one another and also equal to the coefficient for $\hat{\mu}$. In addition, the totals of the RHM's for all equations other than the g_i equal the grand total $Y_{..}$. The total of the RHM's for the g_i equations is equal to $2Y_{..}$. Before the coefficient matrix can be inverted to obtain the inverse elements for use in making the appropriate tests of significance, certain restrictions must be imposed on the constant estimates.

Imposing Restrictions on the Constant Estimates

Convenient restrictions that can be imposed on the constant estimates are

$$\sum_i \hat{g}_i = \sum_j \hat{m}_j = \sum_i \hat{c}_{ij} = \sum_j \hat{c}_{ij} = \sum_i^{p-1} \sum_j \hat{c}_{ij} = \sum_i \hat{r}_{ij} = \sum_j \hat{r}_{ij} = \hat{r}_{ij} + \hat{r}_{ji} = 0$$

The restrictions that $\sum_i \hat{g}_i = \sum_j \hat{m}_j = 0$ may be imposed by completing subtractions by rows and columns as explained previously. The subtractions and additions

required to impose the restrictions on the estimates for the specific effects $(\sum_i \hat{c}_{ij} = \sum_j \hat{c}_{ji} = \sum_i^{p-1} \sum_j \hat{c}_{ij} = 0)$ will be given for the specific design of all possible crosses among four lines.

If the numbers in the subclasses concerned with the measurement of specific effects are represented by:

$$\begin{array}{cccc} & n_{12} & n_{13} & n_{14} \\ n_{21} & & n_{23} & n_{24} \\ n_{31} & n_{32} & & n_{34} \\ n_{41} & n_{42} & n_{43} & \end{array}$$

then subtraction of the last column in each row from all other coefficients results in:

$$\begin{array}{cccc} & n_{12}-n_{14} & n_{13}-n_{14} & n_{14}-n_{14} \\ n_{21}-n_{24} & & n_{23}-n_{24} & n_{24}-n_{24} \\ n_{31}-n_{34} & n_{32}-n_{34} & & n_{34}-n_{34} \\ n_{41}-n_{43} & n_{42}-n_{43} & n_{43}-n_{43} & \end{array}$$

Subtraction by row then yields:

$$\begin{array}{ccc} & n_{12}-n_{14}-n_{42}+n_{43} & n_{13}-n_{14} \\ n_{21}-n_{24}-n_{41}+n_{43} & & n_{23}-n_{24} \\ n_{31}-n_{34}-n_{41}+n_{43} & n_{32}-n_{34}-n_{42}+n_{43} & \end{array}$$

Since the specific effect for a particular cross is measured over the two reciprocal crosses, these values must be combined as follows:

$$\begin{array}{cc} n_{12}-n_{14}-n_{42}+n_{43}+n_{21}-n_{24}-n_{41}+n_{43} & n_{13}-n_{14}+n_{31}-n_{34}-n_{41}+n_{43} \\ & n_{23}-n_{24}+n_{32}-n_{34}-n_{42}+n_{43} \end{array}$$

or

$$\begin{array}{cc} n'_{12}-n'_{14}-n'_{24}+2n_{43} & n'_{13}-n'_{14}-n_{34}+n_{43} \\ & n'_{23}-n'_{24}-n_{34}+n_{43} \end{array}$$

where $n'_{12} = n_{12}+n_{21}$, $n'_{14} = n_{14}+n_{41}$, etc.

The third restriction, $\sum_{ij}^{p-1} \sum \hat{c}_{ij} = 0$, results in the following equations:

$$n'_{12}-n'_{14}-n'_{24}+2n_{43}-n'_{23}+n'_{24}+n_{34}-n_{43} = n'_{12}-n'_{14}-n'_{23}+n'_{34} \quad \ldots . (8)$$

and $$n'_{13}-n'_{14}-n_{34}+n_{43}-n'_{23}+n'_{24}+n_{34}-n_{43} = n'_{13}-n'_{14}-n'_{23}+n'_{24} \quad \ldots . (9)$$

The restriction on the equations for the reciprocal crosses that $\sum_i \hat{r}_{ij} = \sum_j \hat{r}_{ji} = 0$ results in subtraction by row and column by the same procedure as followed for the specific effects. The restriction that the reciprocal effects for each cross must sum to zero ($\hat{r}_{ij} + \hat{r}_{ji} = 0$) yields the following equations:

$$n_{12}-n_{14}-n_{42}+n_{43}-n_{21}+n_{24}+n_{41}-n_{43} = n_{12}-n_{14}-n_{42}-n_{21}+n_{24}+n_{41} \quad . . (10)$$

$$n_{13}-n_{14}-n_{31}+n_{34}+n_{41}-n_{43} = n_{13}-n_{14}-n_{31}+n_{34}+n_{41}-n_{43} \quad . . (11)$$

$$n_{23}-n_{24}-n_{32}+n_{34}+n_{42}-n_{43} = n_{23}-n_{24}-n_{32}+n_{34}+n_{42}-n_{43} \quad . . (12)$$

The subtractions and additions outlined in equations 8-12 are those that are completed for the specific combining ability effects and the residual reciprocal effects in the μ equation. The subtraction and additions among the coefficients for these effects in all other least-squares equations follow the same form as given in the above equations. The procedures outlined in these equations are also completed in the RHM's for the specific and residual reciprocal effects, using the appropriate sums in order to obtain the reduced RHM's.

The equations for the subtractions required to impose these restrictions on the specific and residual reciprocal effects will differ depending on the number of lines involved. However, the procedures shown above can be utilized to determine the appropriate equations. After the coefficient matrix has been reduced there will remain a symmetric matrix of order p(p-1), where p = the number of lines included in the study.

Inversion of the Reduced Matrix

The size of the resulting least-squares matrix is normally so large as to make its inversion by means of a desk calculator extremely laborious if not prohibitive. Programs are readily available for the inversion of matrices by high speed computing equipment. Where the size of the matrix dictates such action, partitioning of the matrix may be required.

Computing the Least-Square Means and Standard Errors

The equations may be solved directly, at the same time as the inversion of the matrix, in order to obtain estimates of the constants μ, g_i, m_j, c_{ij}, and r_{ij}. However, it is sometimes more convenient to obtain the constants from the inverse elements and the RHM's of the equations by means of the formula pre-

viously presented,

$$\sum_j C^{ij} Y_j = \hat{c}_i$$

where C^{ij} is the inverse element for the i^{th} row and j^{th} column of the inverse matrix, Y_j is the RHM for the j^{th} row and $\hat{c}_i$ is the estimate of the i^{th} constant. The constant estimates for the model under discussion will include the least-squares mean and the estimates of the general combining ability effects, the maternal effects, the specific combining ability effects and the reciprocal effects expressed as deviations from the least-squares mean.

In order to estimate the standard errors of the mean and other constant estimates it is necessary to estimate the error mean square, $\hat{\sigma}_e^2$, which is given by

$$\hat{\sigma}_e^2 = \frac{1}{n_{..}-p(p-1)} \left[\sum_{ijk}\sum\sum y_{ijk}^2 - R(\mu,g,m,c,r) \right]$$

where $\frac{1}{n_{..}-p(p-1)}$ = degrees of freedom for error,

$\sum_{ijk}\sum\sum y_{ijk}^2$ = total uncorrected sum of squares,

and $R(\mu,g,m,c,r)$ = reduction in sum of squares due to fitting all constants.

The reduction due to fitting all constants, $R(\mu,g,m,c,r)$ can be computed from all constant estimates and the original set of RHM's as follows:

$$R(\mu,g,m,c,r) = \hat{\mu}Y_{..} + \hat{g}_1 (Y_{1.} + Y_{.1}) + \cdot \cdot \cdot \cdot + \hat{r}_{pp} Y_{pp}.$$

The reduction due to fitting all constants can more easily be calculated from the constant estimates obtained from the solution to the reduced matrix together with the reduced RHM's.

The standard error of the mean is obtained from

$$s_{\hat{\mu}} = \sqrt{C^{\mu\mu} \hat{\sigma}_e^2}$$

where $C^{\mu\mu}$ is the diagonal inverse element corresponding to the $\hat{\mu}$ constant. In order to estimate the standard errors for all the remaining constants it is necessary to complete the inverse matrix for the equations that were subtracted out when imposing the restrictions on the constant estimates. Once the matrix has been completed, standard errors for constant estimates, or linear functions of constant estimates, can be calculated as explained in the first two sections.

Computing Sums of Squares for the Analysis of Variance

The sums of squares for the g_i, m_j, c_{ij}, and r_{ij} can be obtained directly using the formula $B'Z^{-1}B$ where Z^{-1} is the inverse of the segment of the matrix

inverse to the variance-covariance matrix corresponding to the particular effects, and B' and B are the constant estimates for these effects. The sums of squares may also be computed indirectly by differences in reductions after obtaining the solutions to other sets of equations.

In this analysis the sums of squares could be gotten as follows:

Source of variation	Sums of squares
General combining ability	$R(\mu,g,m,c,r) - R(\mu,m,c,r)$
Maternal ability	$R(\mu,g,m,c,r) - R(\mu,g,c,r)$
Specific combining ability	$R(\mu,g,m,c,r) - R(\mu,g,m,r)$
Residual reciprocal effects	$R(\mu,g,m,c,r) - R(\mu,g,m,c)$

The sum of squares for error can be computed as

$$\sum_{ijk}\sum\sum y_{ijk}^2 - R(\mu,g,m,c,r)$$

as previously noted. Since all of the variability among the subclasses is accounted for by the constants being fitted, the total reduction in sum of squares can be obtained by calculating

$$\sum_{ij}\sum \frac{Y_{ij}^2}{r_{ij}} = \text{reduction due to fitting all constants.}$$

The degrees of freedom for the model described are calculated as shown below:

Source of Variation	Degrees of Freedom
Total	$n_{..} -1$
General combining ability	$p-1$
Maternal ability	$p-1$
Specific combining ability	$\frac{p(p-3)}{2}$
Residual reciprocal effects	$\frac{p(p-3)}{2} +1$
Error	$n_{..}-p(p-1)$

Thus it can be seen that with only three lines included in the study, there would be no degrees of freedom available for the estimation of specific combining abilities, and only one degree of freedom available for the estimation of residual reciprocal effects.

Upon completion of the analysis of variance, it can be determined which effects are significant and should be retained. If some of the effects do not appear significant, the equations for these effects can be deleted from the model and more efficient estimates of the remaining effects obtained. The inverse of the variance-covariance matrix when one or more sets of effects have been eliminated, C_A^{-1}, is given by

$$C_A^{-1} = C_R^{-1} - RZ^{-1}R'$$

where Z is the segment of the matrix inverse to the variance-covariance matrix corresponding by row and column to the effects being eliminated, R(R') is the segment of the matrix inverse to the variance-covariance matrix associating these effects with the effects to be retained, and C_R^{-1} is the segment corresponding by row and column to the effects being retained.

The new constant estimates of the effects to be retained, B_A, can be obtained by means of the formula

$$B_A = B_2 - RZ^{-1}B_1,$$

where B_2 represents the constant estimates for the effects to be retained, before adjustment, and B_1 represents the constant estimates of the effects being eliminated. The definitions of R and Z are given above. The new constants can also be obtained by omitting the RHM's of the effects being eliminated and multiplying the RHM's by each row of the new matrix inverse. The elements of the new matrix inverse and the new constant estimates are those which would have been obtained had the effects which were eliminated not been included in the original model.

Estimation of Variance Components

If all effects are assumed to exist and the lines are regarded as random, the expectations of the mean squares are as shown below:

Source of Variation	E(MS)
General combining ability	$\sigma_e^2 + k_6\sigma_r^2 + k_7\sigma_c^2 + k_8\sigma_g^2$
Maternal ability	$\sigma_e^2 + k_4\sigma_r^2 + k_5\sigma_m^2$
Specific combining ability	$\sigma_e^2 + k_2\sigma_r^2 + k_3\sigma_c^2$
Residual reciprocal effects	$\sigma_e^2 + k_1\sigma_r^2$
Error	σ_e^2

The k_1, k_3, k_5, and k_8 values can be calculated by the direct procedure explained in the section discussing the two-way classification without interaction. The k_2, k_4, k_6, and k_7 values can be calculated by the indirect procedure explained in the same section. The variance component estimates are computed by setting the above expectations equal to the computed mean squares and solving the resulting equations.

Numerical Example

The computational procedures outlined above for setting up the least-squares equations, imposing restrictions on the equations, obtaining sums of squares and degrees of freedom to complete the analysis of variance and

estimating the components of variance are illustrated with a set of single cross data involving four lines. The subclass numbers, totals, and means are presented in Table 12.

Table 12. Subclass numbers, totals, and means

Line of Sire		Line of Dam 1	2	3	4	Sums
1	Number	22	12	10	12	56
	Total	1231.0	767.2	621.0	787.5	3406.7
	Mean	55.955	63.933	62.100	65.625	
2	Number	12	26	10	10	58
	Total	712.0	1725.0	597.4	610.5	3644.9
	Mean	59.333	66.346	59.740	61.050	
3	Number	12	12	12	12	48
	Total	771.3	773.9	645.5	799.9	2990.6
	Mean	64.275	64.492	53.792	66.658	
4	Number	14	8	12	12	46
	Total	860.8	459.2	605.4	731.6	2657.0
	Mean	61.486	57.400	50.450	60.967	
Sums	Number	60	58	44	46	208
	Total	3575.1	3725.3	2469.3	2929.5	12699.2

These data are the 28-day weights of rats resulting from an experiment by Kidwell, et al.[14] The observations are the sex by litter means and only those litters with rats of both sexes are included in the analysis. Therefore, the sex effects are orthogonal to the subclass differences. The least-squares coefficient matrix, using the model previously presented in this section, is shown in Table 13. It can be observed that the segment corresponding to the rows and columns to the general combining ability is filled in completely, whereas the segments corresponding by row and column to the other effects have entries only on the diagonals. In addition, the RHM's for the general combining ability do not equal the sum for the μ equation.

The least-squares coefficient matrix is then reduced by imposing the restrictions previously presented and this reduced matrix is shown in Table 14.

If the equations are solved at the same time the matrix is inverted, by including the RHM's in the equations, then the constant estimates are obtained directly. If the solution to the equations is not computed directly, the constants can be calculated from the inverse elements and the RHM's of the equations, since $\sum_j C^{ij} Y_j = \hat{c}_i$

as noted previously.

14/ Kidwell, J. F., H. J. Weeth, W. R. Harvey, L. H. Haverland, C. E. Shelby, and R. T. Clark. Heterosis in crosses of inbred lines of rats. Genetics 45:225-231. 1960.

Table 13. Least-squares equations for the numerical example under the model $y_{ijk} = \mu + g_i + g_j + m_j + c_{ij} + r_{ij} + e_{ijk}$

	$\hat{\mu}$	$\hat{g}_1$	$\hat{g}_2$	$\hat{g}_3$	$\hat{g}_4$	$\hat{m}_1$	$\hat{m}_2$	$\hat{m}_3$	$\hat{m}_4$	$\hat{c}_{12}$	$\hat{c}_{13}$	$\hat{c}_{14}$	$\hat{c}_{23}$	$\hat{c}_{24}$	$\hat{c}_{34}$	$\hat{r}_{12}$	$\hat{r}_{13}$	$\hat{r}_{14}$	$\hat{r}_{21}$	$\hat{r}_{23}$	$\hat{r}_{24}$	$\hat{r}_{31}$	$\hat{r}_{32}$	$\hat{r}_{34}$	$\hat{r}_{41}$	$\hat{r}_{42}$	$\hat{r}_{43}$	RHM
μ:	136	72	64	68	68	38	32	32	34	24	22	26	22	18	24	12	10	12	12	10	10	12	12	12	14	8	12	8366.1
g_1:	72	72	24	22	26	38	12	10	12	24	22	26				12	10	12	12			12			14			4519.8
g_2:	64	24	64	22	18	12	32	10	10	24			22	18		12			12	10	10		12			8		3920.2
g_3:	68	22	22	68	24	12	12	32	12		22		22		24		10			10		12	12	12			12	4168.9
g_4:	68	26	18	24	68	14	8	12	34			26		18	24			12			10			12	14	8	12	4123.3
m_1:	38	38	12	12	14	38				12	12	14							12			12			14			2344.1
m_2:	32	12	32	12	8		32			12			12	8		12							12			8		2000.3
m_3:	32	10	10	32	12			32			10		10		12		10			10							12	1823.8
m_4:	34	12	10	12	34				34			12		10	12			12			10			12				2197.9
c_{12}:	24	24	24			12	12			24						12			12									1479.2
c_{13}:	22	22		22		12		10			22						10					12						1392.3
c_{14}:	26	26			26	14			12			26						12							14			1648.3
c_{23}:	22		22	22			12	10					22							10			12					1371.3
c_{24}:	18		18		18		8		10					18							10					8		1069.7
c_{34}:	24			24	24			12	12						24									12			12	1405.3
r_{12}:	12	12	12				12			12						12												767.2
r_{13}:	10	10		10				10			10						10											621.0
r_{14}:	12	12			12				12			12						12										787.5
r_{21}:	12	12	12			12				12									12									712.0
r_{23}:	10		10	10				10					10							10								597.4
r_{24}:	10		10		10				10			0		10							10							610.5
r_{31}:	12	12		12		12					12											12						771.3
r_{32}:	12		12	12			12						12										12					773.9
r_{34}:	12			12	12				12						12									12				779.9
r_{41}:	14	14			14	14						14													14			860.8
r_{42}:	8		8		8		8							8												8		459.2
r_{43}:	12			12	12			12							12												12	605.4

Table 14. Reduced least-squares equations and RHM's.

	$\hat{\mu}$	$\hat{g}_1$	$\hat{g}_2$	$\hat{g}_3$	$\hat{m}_1$	$\hat{m}_2$	$\hat{m}_3$	$\hat{c}_{12}$	$\hat{c}_{13}$	$\hat{r}_{12}$	$\hat{r}_{13}$	$\hat{r}_{23}$	RHM
μ :	136	4	-4	0	4	-2	-2	0	-8	4	0	-4	8366.1
g_1 :	4	88	48	40	46	26	20	0	4	-2	-2	2	396.5
g_2 :	-4	48	96	48	22	48	22	4	4	-2	-2	-2	-203.1
g_3 :	0	40	48	88	20	26	42	4	8	-4	-4	0	45.6
m_1 :	4	46	22	20	72	34	34	-2	0	4	2	-2	146.2
m_2 :	-2	26	48	26	34	66	34	0	-2	6	0	-6	-197.6
m_3 :	-2	20	22	42	34	34	66	2	2	2	-2	-4	-374.1
c_{12}:	0	0	4	4	-2	0	2	96	48	-2	-2	2	-135.1
c_{13}:	-8	4	4	8	0	-2	2	48	88	0	-4	0	-557.6
r_{12}:	4	-2	-2	-4	4	6	2	-2	0	68	26	-18	279.8
r_{13}:	0	-2	-2	-4	2	0	-2	-2	-4	26	72	24	117.5
r_{23}:	-4	2	-2	0	-2	-6	-4	2	0	-18	24	64	-133.3

The inverse elements for the above matrix are presented in Table 15, together with the constant estimates. The total reduction in sum of squares due to fitting all constants can be obtained by multiplying the column vector of all RHM's by the row vector of all constant estimates. The same result is obtained by using the constants estimated from the reduced matrix and the corresponding reduced RHM's. For example,

$$R(\mu,g,m,c,r) = (61.378)(8366.1)+(2.941)(396.5)+ \ + \ + (1.229)(-133.3)= 517,127$$

This reduction in sum of squares can also be obtained from the subclass numbers and totals given in Table 12, as

$$R(\mu,g,m,c,r) = \frac{(767.2)^2}{12} + \frac{(621.0)^2}{10} + \ + \ + \ \frac{(605.4)^2}{12} = 517,131$$

Since only three significant digits were carried in some of the constant estimates the two answers do not agree exactly.

The next step is to calculate the error sum of squares. Normally this is obtained in a least-squares analysis from

$$\sum_{ijk}\sum\sum y_{ijk}^2 - R(\text{all constants}).$$

However, in these data the sex and sex by subclass effect would remain in the error. The sum of squares for error, therefore, was obtained by also subtracting the corrected sums of squares for sex and sex by subclass from the total uncorrected sum of squares. The resulting error sum of squares, with 136-24 = 122 degrees of freedom, was found to be 5,786.3.

Since the differences among the $\hat{r}_{ij}$ were small, and there is little biological basis to expect these effects to exist, a test of significance was made

Table 15. Inverse matrix of the variance-covariance matrix ($x10^{-6}$).

	$\hat{\mu}$	$\hat{g}_1$	$\hat{g}_2$	$\hat{g}_3$	$\hat{m}_1$	$\hat{m}_2$	$\hat{m}_3$	$\hat{c}_{12}$	$\hat{c}_{13}$	$\hat{r}_{12}$	$\hat{r}_{13}$	$\hat{r}_{23}$	Constant Estimates
μ :	7498	-434	645	-484	-595	174	694	-554	1009	-384	50	434	61.378
g_1 :	-434	25651	-9375	-6901	-16667	4948	5729	434	-347	-391	1302	-1450	2.941
g_2 :	645	-9375	25595	-8519	6101	-17708	5729	-645	-124	1488	-856	781	-1.294
g_3 :	-484	-6901	-8519	24163	5580	4427	-15625	484	-1339	577	186	-391	2.753
m_1 :	-595	-16667	6101	5580	31548	-10417	-11458	595	-446	149	-1414	521	-1.641
m_2 :	174	4948	-17708	4427	-10417	35937	-11458	-174	1389	-2344	521	781	1.426
m_3 :	694	5729	5729	-15625	-11458	-11458	33333	-694	347	0	521	521	-5.784
c_{12} :	-554	434	-645	484	595	174	-694	14443	-7953	384	-50	-434	-1.285
c_{13} :	1009	-347	-124	-1339	-446	1389	347	-7953	16005	-918	992	-174	-.172
r_{12} :	-384	-391	1488	577	149	-2344	0	384	-918	22080	-11384	10286	.767
r_{13} :	50	1302	-856	186	-1414	521	521	-50	992	-11384	21949	-11458	.984
r_{23} :	434	-1450	781	-391	521	781	521	-434	-174	10286	-11458	23047	1.229

to determine whether this variable could be deleted. The adjusted sum of squares for R, the residual reciprocal effects, is obtained from the general formula $B'Z^{-1}B$ as follows:

$$\text{S. Sqs.--R} = \begin{bmatrix} .767 & .984 & 1.229 \end{bmatrix} (10^{-6}) \begin{bmatrix} 22080 & -11384 & 10286 \\ -11384 & 21949 & -11458 \\ 10286 & -11458 & 23047 \end{bmatrix}^{-1} \begin{bmatrix} .767 \\ .984 \\ 1.229 \end{bmatrix}$$

$$= \begin{bmatrix} .767 & .984 & 1.229 \end{bmatrix} \begin{bmatrix} 66.3559 & 25.6001 & -16.8877 \\ 25.6001 & 71.4053 & 24.0742 \\ -16.8877 & 24.0742 & 62.8954 \end{bmatrix} \begin{bmatrix} .767 \\ .984 \\ 1.229 \end{bmatrix}$$

$$= \begin{bmatrix} 55.3305 & 119.4853 & 88.0346 \end{bmatrix} \begin{bmatrix} .767 \\ .984 \\ 1.229 \end{bmatrix}$$

$$= 268.21$$

The test for the significance of differences among the $\hat{r}_{ij}$ is then completed as follows:

Source of Variation	d.f.	Sum of Squares	Mean Square	F
Residual reciprocal effects	3	268.2	89.4	1.89
Error	122	5786.3	47.4	

Since the residual reciprocal effects were non-significant they were deleted from the analysis in order to obtain more efficient estimates of μ, g_i, m_j, and s_{ij} the inverse elements of Table 14 were adjusted by use of the formula

$$C_A^{-1} = C_R^{-1} - RZ^{-1}R' \ .$$

The $RZ_R^{-1}R'$ matrix, which was subtracted from the equations in the segment corresponding to the remaining effects is presented in Table 16.

Table 16. The $RZ_R^{-1}R'$ matrix used to delete the r_{ij}. (x10^6)

	$\hat{\mu}$	$\hat{g}_1$	$\hat{g}_2$	$\hat{g}_3$	$\hat{m}_1$	$\hat{m}_2$	$\hat{m}_3$	$\hat{c}_{12}$	$\hat{c}_{13}$
μ :	28	-33	-23	-31	8	103	20	-28	27
g_1 :	-33	127	-35	48	-91	-91	-3	33	28
g_2 :	-23	-35	101	35	17	-149	0	23	-63
g_3 :	-31	48	35	44	-35	-130	6	31	-22
m_1 :	8	-91	17	-35	112	35	-46	-8	-51
m_2 :	103	-91	-149	-130	35	441	51	-103	121
m_3 :	20	-3	0	6	-46	51	50	-20	37
c_{12}:	-28	33	23	31	-8	-103	-20	28	-27
c_{13}:	27	28	-63	-22	-51	121	37	-27	68

These values were subtracted element-by-element from those in the segment for the remaining effects, resulting in the new inverse matrix of the variance-covariance matrix with the r_{ij} effects deleted. This matrix is presented in Table 17.

Table 17. The inverse matrix (C_A^{-1}) of the variance-covariance matrix with the r_{ij} effects deleted. $(\times 10^{-6})$

	$\hat{\mu}$	$\hat{g}_1$	$\hat{g}_2$	$\hat{g}_3$	$\hat{m}_1$	$\hat{m}_2$	$\hat{m}_3$	$\hat{c}_{12}$	$\hat{c}_{13}$
μ :	7470	-401	668	-453	-603	71	674	-526	982
g_1 :	-401	25524	-9340	-6949	-16576	5039	5732	401	-375
g_2 :	668	-9340	25494	-8554	6084	-17559	5729	-668	-61
g_3 :	-453	-6949	-8554	24119	5615	4557	-15619	453	-1317
m_1 :	-603	-16576	6084	5615	31436	-10452	-11412	603	-395
m_2 :	71	5039	-17539	4557	-10452	35496	-11509	277	1268
m_3 :	674	5732	5729	-15619	-11412	-11509	33283	-674	310
c_{12} :	-526	401	-668	453	603	277	-674	14415	-7926
c_{13} :	982	-375	61	-1317	-395	1268	310	-7926	15937

Since the r_{ij} were assumed to equal zero, these RHM's were omitted and the new constant estimates obtained as before, by multiplying the RHM's by each row of the new inverse. These constants are shown below:

$$\hat{\mu} = 10^{-6}\left[(7470)(8366.1) + (-401)(396.5) + \quad + \quad + (982)(-557.6)\right] = 61.35$$

$\hat{g}_1 = 2.94$ $\hat{m}_1 = -1.53$ $\hat{c}_{12} = -1.33$

$\hat{g}_2 = -1.34$ $\hat{m}_2 = 1.38$ $\hat{c}_{13} = -.22$

$\hat{g}_3 = 2.73$ $\hat{m}_3 = -5.89$

The remaining constants ($\hat{g}_4, \hat{m}_4, \hat{c}_{14}, \hat{c}_{23}, \hat{c}_{24}, \hat{c}_{34}$) can be obtained by differences, since $\hat{g}_1+\hat{g}_2+\hat{g}_3+\hat{g}_4 = \hat{m}_1+\hat{m}_2+\hat{m}_3+\hat{m}_4 = 0$ and the $\hat{c}_{ij}$ sum to zero by row and column and for $\sum\limits_{ij}^{p-1}\sum \hat{c}_{ij}$.

The reduction in sum of squares due to fitting $\mu, g, m,$ and c is then calculated by multiplying the reduced RHM's by the corresponding constant estimates, as follows

$$R(\mu,g,m,c) = (61.35)(8366.1)+(2.94)(396.5) + + + (.22)(-557.6) = 516,472.4$$

Normally, this reduction in sum of squares would be subtracted from the total uncorrected sum of squares to obtain a new error term. However, since a difference in sexes was also involved in this analysis, the error term computed from among the litters within the sex by cross subclasses was used.

Completion of the Analysis of Variance

In order to complete the analysis of variance, it is only necessary to compute the sums of squares for the general combining ability effects, maternal effects, and the specific combining ability effects. The sum of squares for the general combining ability effects is computed as follows:

$$\text{S. Sqs.---G} = \begin{bmatrix} 2.94 & -1.34 & 2.73 \end{bmatrix} \begin{bmatrix} .025524 & -.009340 & -.006949 \\ -.009340 & .025494 & -.008554 \\ -.006949 & -.008554 & .024119 \end{bmatrix}^{-1} \begin{bmatrix} 2.94 \\ -1.34 \\ 2.73 \end{bmatrix}$$

$$= \begin{bmatrix} 2.94 & -1.34 & 2.73 \end{bmatrix} \begin{bmatrix} 57.7447 & 30.3487 & 27.4003 \\ 30.3487 & 60.4735 & 30.1920 \\ 27.4003 & 30.1920 & 60.0631 \end{bmatrix} \begin{bmatrix} 2.94 \\ -1.34 \\ 2.73 \end{bmatrix}$$

$$= \begin{bmatrix} 203.9050 & 90.6148 & 204.0719 \end{bmatrix} \begin{bmatrix} 2.94 \\ -1.34 \\ 2.73 \end{bmatrix}$$

$$= 1035.2$$

The sum of squares for the maternal effects is computed in a similar fashion:

$$\text{S. Sqs.---M} = \begin{bmatrix} -1.53 & 1.38 & -5.89 \end{bmatrix} \begin{bmatrix} .031436 & -.010452 & -.011412 \\ -.010452 & .035496 & -.011509 \\ -.011412 & -.011509 & .033283 \end{bmatrix}^{-1} \begin{bmatrix} -1.53 \\ 1.38 \\ -5.89 \end{bmatrix}$$

$$= \begin{bmatrix} -1.53 & 1.38 & -5.89 \end{bmatrix} \begin{bmatrix} 47.7436 & 21.8113 & 23.9123 \\ 21.8113 & 41.6942 & 21.8964 \\ 23.9123 & 21.8964 & 45.8160 \end{bmatrix} \begin{bmatrix} -1.53 \\ 1.38 \\ -5.89 \end{bmatrix}$$

$$= \begin{bmatrix} -183.7916 & -104.8031 & -276.2250 \end{bmatrix} \begin{bmatrix} -1.53 \\ 1.38 \\ -5.89 \end{bmatrix}$$

$$= 1763.5$$

The sum of squares for specific combining ability is computed similarly:

$$\text{S. Sqs.---C} = \begin{bmatrix} -1.33 & -.22 \end{bmatrix} \begin{bmatrix} .014415 & -.007926 \\ -.007926 & .015937 \end{bmatrix}^{-1} \begin{bmatrix} -1.33 \\ -.22 \end{bmatrix}$$

$$= \begin{bmatrix} -1.33 & -.22 \end{bmatrix} \begin{bmatrix} 95.4826 & 47.4867 \\ 47.4867 & 86.3639 \end{bmatrix} \begin{bmatrix} -1.33 \\ -.22 \end{bmatrix}$$

$$= \begin{bmatrix} 137.4389 & 82.1574 \end{bmatrix} \begin{bmatrix} -1.33 \\ -.22 \end{bmatrix}$$

$$= 200.9$$

The completed analysis of variance is presented below:

Source of Variation	d.f.	Sum of Squares	Mean Squares
General combining abilities (G)	3	1035.2	345.1
Maternal abilities (M)	3	1763.5	587.8**
Specific combining abilities (C)	2	200.9	100.4
Error	122	5786.3	47.4

Since the specific combining ability effects are not significant, these equations could be deleted from the model as were the equations for the sex-linkage effects. Assuming the specific combining ability and the sex-linkage effects to be zero, the mean square for error is the appropriate term for testing the significance of the general combining ability effects as well as the maternal ability effects when a random model is considered. Should the line effects be considered fixed, as in crossing breeds of cattle where similar samples of the breeds could be selected for repeated experiments, the mean square from the error line would be appropriate for testing all effects.

Estimation of Variance Components

Variance component estimates can be calculated readily in this analysis by making use of the inverse elements of the segments which were inverted to obtain the sums of squares to calculate the coefficients for the variance components. The coefficient for the major variance component in each line of the analysis of variance is computed by means of the general formula

$$k = \frac{\sum_{i} Z^{ii} - \frac{1}{d.f.} \sum\sum_{ij,\ i \neq j} Z^{ij}}{m}$$

presented in the section on the two-way classification without interaction, and again not to be used for interaction variance components.

The expectations of the mean squares are shown below.

Source of variation	d.f.	E(MS)
General combining abilities (G)	3	$\sigma_e^2 + k_3\ \sigma_c^2 + k_4\ \sigma_g^2$
Maternal abilities (M)	3	$\sigma_e^2 + k_2\ \sigma_m^2$
Specific combining abilities (C)	2	$\sigma_e^2 + k_1\ \sigma_c^2$
Error	122	σ_e^2

Utilizing the inverse elements of the matrix which was used to calculate the sum of squares for specific effects,

$$k_1 = \frac{181.8465 - 47.4867}{6} = 22.39$$

and $$\hat{\sigma}_c^2 = \frac{100.4 - 47.4}{22.39} = 2.4 \ .$$

Performing the same calculations with the inverse elements of the matrix used to calculate the sum of squares for maternal effects,

$$k_2 = \frac{135.2538 - 45.0800}{4} = 22.54$$

and $$\hat{\sigma}_m^2 = \frac{587.8 - 47.4}{22.54} = 24.0 \ .$$

The inverse elements used in the computation of the sum of squares for the general combining abilities are then used to calculate k_4, as

$$k_4 = \frac{178.2813 - 58.6273}{4} = 29.91 \ .$$

The coefficient for the component of σ_c^2 appearing in the E(MS) for the general combining abilities, k_3, may also be determined from the matrix used to compute the sum of squares for general combining abilities, when all subclasses are filled. The formula for calculating this value in this case is

$$k' = \frac{\sum_i Z^{ii} - \frac{1}{d.f.} \sum\sum_{ij,\ i\neq j} Z^{ij}}{m(m-2)}$$

Utilizing this formula,

$$k_3 = \frac{178.2813 - 58.6273}{8} = 14.96$$

and $$\hat{\sigma}_g^2 = \frac{345.1 - 47.4 - (14.96)(2.4)}{29.91} = 8.8$$

Mean Separation

Since significant differences were found among the maternal effects, mean separation procedures, such as Duncan's Multiple Range Test, can be applied to determine which means differ significantly from one another. If the lines or breeds were considered as fixed, significant differences would also have existed among the general combining ability effects. Since the mean separation

procedures are the same for all effects, the test will be used only for the general combining abilities. The computational details regarding this test were discussed in both the first and second sections, so the results of this test are presented in Table 18 without further explanation.

Table 18. Mean separation for general combining ability effects with Duncan's Multiple Range Test (.05 level)

Comparison	$(\bar{y}_i-\bar{y}_j)$	$\sqrt{\frac{2}{C^{ii}+C^{jj}-2C^{ij}}}$	Product differences	$\hat{\sigma}^2_e\, z p n_2$
g_1 vs g_3	.21	5.610	1.18	19.28
g_1 vs g_2	4.28*	5.357	22.93	20.31
g_1 vs g_4	7.27*	5.366	39.01	21.00
g_3 vs g_2	4.07*	5.475	22.28	19.28
g_3 vs g_4	7.06*	5.472	38.63	20.31
g_2 vs g_4	2.99	5.499	16.44	19.28

Use of the Transformation Matrix

The writing out of the coefficient matrix, the subsequent subtractions performed in imposing restrictions on the constant estimates, and the final inversion of the matrix is a time-consuming process. Much of the work can be eliminated if a transformation matrix is available, or can be written out quickly, as is possible with many designs. Advantage can then be taken of the fact that the inverse of the matrix of subclass numbers (a diagonal matrix) can be obtained readily as this is nothing more than the reciprocals of the subclass numbers. In addition, the constants are merely the subclass means. Transformation of the inverse of the subclass numbers then yields the inverse matrix that would have been obtained under the methods outlined in this section. This transformation is accomplished by

$$KD^{-1}K'$$

where K = the transformation matrix, D^{-1} = the inverse to the matrix of subclass numbers, and K' is the transpose of the transformation matrix. Transformation of the constant estimates (the subclass means) then yields the constant estimates that would have been obtained by the methods of this section. Transformation of the subclass constant estimates is accomplished by

$$KB$$

where K is the transformation matrix and B is the vector of subclass means.

The transformation matrix for the model under discussion and the general formulas for writing out the transformation matrices for single cross data involving more than four lines, when the linebreds have not been included, have

been given by Harvey.[15] The transformation matrix for the model under discussion, the subclass means, the reciprocals and the transformed constant estimates are presented in Table 19.

Table 19. The transformation matrix (K), the subclass means the reciprocals of the subclass numbers, and the transformed constant estimates.

	s_{12}	s_{13}	s_{14}	s_{21}	s_{23}	s_{24}	s_{31}	s_{32}	s_{34}	s_{41}	s_{42}	s_{43}	Means	Reciprocals	Constant Estimates
μ	2	2	2	2	2	2	2	2	2	2	2	2	63.933	.0833333	61.378
g_1	6	6	6	0	-3	-3	0	-3	-3	0	-3	-3	62.100	.1000000	2.941
g_2	0	-3	-3	6	6	6	-3	0	-3	-3	0	-3	65.625	.0833333	-1.294
g_3	-3	0	-3	-3	0	-3	6	6	6	-3	-3	0	59.333	.0833333	2.753
m_1 $\frac{1}{24}$	-6	-6	-6	6	0	0	6	0	0	6	0	0	59.740	.1000000	-1.641
m_2	6	0	0	-6	-6	-6	0	6	0	0	6	0	61.050	.1000000	1.426
m_3	0	6	0	0	6	0	-6	-6	-6	0	0	6	64.275	.0833333	-5.784
c_{12}	4	-2	-2	4	-2	-2	-2	-2	4	-2	-2	4	64.492	.0833333	-1.285
c_{13}	-2	4	-2	-2	-2	4	4	-2	-2	-2	4	-2	66.658	.0833333	-.172
r_{12}	6	-3	-3	-6	3	3	3	-3	0	3	-3	0	61.486	.0714286	.767
r_{13}	-3	6	-3	3	-3	0	-6	3	3	3	0	-3	57.400	.1250000	.984
r_{23}	3	-3	0	-3	6	-3	3	-6	3	0	3	-3	50.450	.0833333	1.229

The transformed constant estimates shown in the last column of table 8 were computed as follows:

$$\hat{\mu} = \frac{(2)(63.933) + (2)(62.100) + + + (2)(50.450)}{24} = 61.378$$

$$\hat{g}_1 = \frac{(6)(63.933) + (6)(62.100) + + - (3)(50.450)}{24} = 2.941$$

$$\vdots$$

$$\hat{r}_{23} = \frac{(3)(63.933) - (3)(62.100) + + - (3)(50.450)}{24} = 1.229$$

These constant estimates are the same as those presented in table 4 and thus

[15] Harvey, Walter R. Analysis of data with unequal subclass numbers when a set of orthogonal comparisons among the subclass means is desired. Paper presented in the Biometrics Society Section of the AIBS meetings at Pennsylvania State University, August 31, 1959.

are the same as would have been obtained by the methods outlined previously in this section. The inverse elements are obtained as follows:

$$C^{\mu\mu} = \frac{(2)^2(.0833333)+(2)^2(.1000000) + \; + \; +(2)^2(.0833333)}{(24)^2} = .007498$$

$$C^{\mu g_1} = \frac{(2)(6)(.0833333)+(2)(6)(.1000000)+ \; + \; - (2)(3)(.0833333)}{(24)^2} = -.000434$$

$$C^{g_1 g_1} = \frac{(6)^2(.0833333)+(6)^2(.1000000) + \; + \; + \; (-3)^2(.0833333)}{576} = .025651$$

$$C^{g_1 g_1} = \frac{(6)(-3)(.1000000)-(6)(3)(.0833333) + \; + \; +(3)(3)(.0833333)}{576} = -.009375$$

etc.

While the transformed inverse matrix C^{-1} can be calculated on a desk calculator, these matrix multiplications can be completed readily on high speed computers, saving considerable time. Upon completion of the inverse matrix, the analysis would proceed as outlined.

SIMULTANEOUS ANALYSIS OF DATA FROM CROSSES AND PUREBREDS I

When it is possible to include linebreds or purebreds in the design of a single cross experiment an estimate of heterosis is then available. The linebred effects may be estimated independently of or simultaneously with the general combining ability and maternal effects. The analysis to be considered in this section includes the linebreds in the experiment but estimates these effects independently of the general combining ability and maternal effects.

Model

The mathematical model underlying this analysis is

$$y_{hijk} = \mu + a_h + p_{1ii} + g_{2i} + g_{2j} + m_{2j} + c_{2ij} + r_{2ij} + e_{hijk} \quad . \; . \; . \; . \; (8)$$

where y_{hijk} = the k^{th} observation on the progeny of a mating of a dam from the j^{th} line with a sire of the i^{th} line in the h^{th} type of breeding (purebred or crossbred); μ = the over-all mean when equal subclass frequencies exist and there are equal frequencies of linebreds and crossbreds; a_h = an effect common to all progeny of the h^{th} type of breeding (linebred or crossbred), the difference between these effects being an estimate of heterosis; and p_{1ii} = an effect common to all progeny of a mating between a dam of the i^{th} line and

a sire of the i^{th} line. The remaining symbols represent the same effects as described in model(7). The subscript 1 denotes that the effects are measured only among the progeny of the first type of mating (linebred) and the subscript 2 denotes that the effects are measured only among the progeny of the second type of mating (crossbred).

Least-Squares Equations

The least-squares equations for this model are presented in Table 20. It can be seen from these equations that while the sums of the coefficients for the heterosis effects equal the coefficient for $\hat{\mu}$ in the μ equation, none of the sums of coefficients for the other effects equal the coefficient for $\hat{\mu}$. Also, in the sums making up the RHM's, only the sum of the RHM's for the heterotic effects equals the sum for the μ equation. Since the linebreds are not used in estimating the general combining abilities or maternal abilities, and are not involved in estimating specific or sex-linked effects, several cells in the table contain no entries. Again, it is noted that in the segment of the coefficient matrix corresponding by row and column to the general combining ability effects, there are entries in the off-diagonal elements. In the segments corresponding to the other effects, only zeros appear in the off-diagonals.

Imposing Restrictions on the Constant Estimates

The restrictions imposed on the constant estimates with this model are similar to those of model (7) except that the type-of-breeding effects and the purebred effects must be taken into account. While many restrictions are possible, the following are convenient for this analysis:

$$\sum_h \hat{a}_h = \sum_i \hat{p}_{1ii} = \sum_i \hat{g}_{2i} = \sum \hat{m}_{2j} = \sum_i \hat{c}_{ij} = \sum_j \hat{c}_{ji} = \sum_{ij}^{p-1}\sum \hat{c}_{ij} = \sum_i \hat{r}_{ij} = \sum_j \hat{r}_{ji} = \hat{r}_{ij} + \hat{r}_{ji} = 0.$$

These restrictions force the constant estimates for type-of-breeding-effects to sum to zero about $\hat{\mu}$, the constant estimates for purebred effects to sum to zero about $\hat{\mu} + \hat{a}_1$, and the remaining constant estimates to sum to zero about $\hat{\mu} + \hat{a}_2$. The restrictions on the constant estimates for the $\hat{a}_h$ and $\hat{p}_{1ii}$ are effected by subtraction by row and column as discussed in the first section. The remaining restrictions are imposed by the same procedures as outlined for model (7).

Inversion of the Reduced Matrix

After the restrictions discussed above have been imposed on the constant estimates, there will remain a symmetric matrix of order p^2, where p = the number of lines included. Inversion of a matrix of this order is of course most readily accomplished on a high speed computer.

Table 20. Least-squares Equations for Model (8).

	$\hat{\mu}$	$\hat{a}_h$	$\hat{p}_{1ii}$	$\hat{g}_{2i}$	$\hat{m}_{2j}$	$\hat{c}_{2ij}$	$\hat{r}_{2ij}$	RHM
μ	$n_{...}$	$n_{h..}$	n_{1ii}	$n_{2i.}+n_{2.i}$	$n_{2.j}$	$n_{2ij}+n_{2ji}$	n_{2ij}	$Y_{...}$
a_h	$n_{h..}$	${}_{C}n_{j..}{}^{C}$	n_{1ii}	$n_{2i.}+n_{2.i}$	$n_{2.j}$	$n_{2ij}+n_{2ji}$	n_{2ij}	$Y_{h..}$
p_{1ii}	$n_{1..}$	$n_{1..}$	${}_{O}n_{1..}{}^{C}$	-	-	-	-	Y_{1ii}
g_{qi}	$n_{2i.}+n_{2.i}$	$n_{2i.}+n_{2.i}$	-	$n_{2ij}+n_{2ji}$ $n_{2i.}+n_{2.i}$ $n_{2ij}+r_{2ji}$	$n_{2.j}$	$n_{2ij}+n_{2ji}$	n_{2ij}	$Y_{2i.}+Y_{2.i}$
m_{2j}	$n_{2.j}$	$n_{2.j}$	-	$n_{2.j}$	${}_{O}n_{2.j}{}^{O}$	n_{2ij}	n_{2ij}	$Y_{2.j}$
c_{2kj}	$n_{2ij}+n_{2ji}$	$n_{2ij}+n_{2ji}$	-	$n_{2ij}+n_{2ji}$	n_{2ij}	${}_{O}n_{2ij}+n_{2ji}{}^{O}$	n_{2ij}	$Y_{2ij}+Y_{2ji}$
r_{2ij}	n_{2ij}	n_{2ij}	-	n_{2ij}	n_{2ij}	n_{2ij}	${}_{O}n_{2ij}{}^{O}$	Y_{2ij}

Computing the Least-Squares Means and Standard Errors (Fixed Model)

The constant estimates for the least-squares mean and all effects included in the model can be obtained by solving the equations together with the inversion of the variance-covariance matrix. If this is not done, then the constants can be estimated from the inverse elements and the RHM's of the equations by

$$\sum_j C^{ij} Y_j = \hat{c}_i$$

where C^{ij} is the inverse element for the i^{th} row and j^{th} column of the matrix inverse, Y_j is the RHM for the j^{th} row and $\hat{c}_i$ is the estimate of the i^{th} constant. Since the constant estimate for the type-of-breeding effect is determined as a deviation from $\hat{\mu}$, the least-squares mean for the purebreds is given by

$$\hat{\mu} + \hat{a}_1$$

and the least-squares mean for linecrosses or crossbreds is given by

$$\hat{\mu} + \hat{a}_2$$

The least-squares mean for a purebred group is then

$$\hat{\mu} + \hat{a}_1 + \hat{p}_{1ii}$$

and the least-squares mean for a linecross group is

$$\hat{\mu} + \hat{a}_2 + \hat{g}_{2i} + \hat{g}_{2j} + \hat{m}_{2j} + \hat{c}_{2ij} + \hat{r}_{2ij} .$$

The standard errors for these means are determined as follows:

$$s_{\hat{\mu}} = \sqrt{C^{\mu\mu}} \; \hat{\sigma}_e$$

$$s_{(\hat{\mu}+\hat{a}_1)} = \sqrt{C^{\mu\mu} + C^{a_1a_1} + 2C^{\mu a_1}} \; \hat{\sigma}_e$$

$$s_{(\hat{\mu}+\hat{a}_2)} = \sqrt{C^{\mu\mu} + C^{a_2a_2} + 2C^{\mu a_2}} \; \hat{\sigma}_e$$

$$s_{(\hat{\mu}+\hat{a}_1+\hat{p}_{1ii})} = \sqrt{C^{\mu\mu} + C^{a_1a_1} + C^{p_{1ii}} + 2C^{\mu a_1} + 2C^{\mu p_{1ii}} + 2C^{a_1 p_{1ii}}} \; \hat{\sigma}_e$$

$$s_{(\hat{\mu}+\hat{a}_2+\hat{g}_{2i}+\hat{g}_{2j}+\hat{m}_{2j}+\hat{c}_{2ij}+\hat{r}_{2ij})} = (C^{\mu\mu} + C^{a_2a_2} + C^{g_{2i}g_{2i}} + C^{g_{2j}g_{2j}}$$

$$+ C^{c_{2ij}c_{2ij}} + C^{r_{2ij}r_{2ij}} + 2C^{\mu a_2} + 2C^{\mu g_{2i}} + 2C^{\mu g_{2j}} + 2C^{\mu m_{2j}} + 2C^{\mu c_{2ij}}$$

$$+ 2C^{\mu r_{2ij}} + 2C^{a_2 g_{2i}} + 2C^{a_2 g_{2j}} + 2C^{a_2 m_{2j}} + 2C^{a_2 c_{2ij}} + 2C^{a_2 r_{2ij}}$$

$$= 2C^{g_{2i}g_{2j}} + 2C^{g_{2i}m_{2j}} + 2C^{g_{2i}c_{ij}} + 2C^{g_{2i}r_{2ij}} + 2C^{g_{2j}m_{2j}} + 2C^{g_{2j}c_{2ij}}$$

$$+ 2C^{g2j r2ij} + 2C^{m2j c2ij} + 2C^{m2j r2ij} + 2C^{c2ij r2ij})^{\frac{1}{2}} \quad \hat{\sigma}_e$$

where the C's denote elements of the matrix inverse to the variance-covariance matrix, the superscripts designate the particular elements and σ_e is the error standard deviation. While the standard errors for the over-all mean, purebreds, crossbreds and purebred groups can be obtained readily, it is obvious that considerable effort is involved in obtaining the standard errors for the crossbred groups by the use of the formula given above. A shorter method of obtaining these constants will be presented later.

The error variance, $\hat{\sigma}_e^2$, is normally obtained by the following formula

$$\hat{\sigma}_e^2 = \frac{\sum_{hijk}\sum\sum\sum y_{hijk}^2 - R(\mu,a,g,m,c,r)}{\text{d.f. for error}}$$

where $\sum_{hijk}\sum\sum\sum y_{hijk}^2$ = the total uncorrected sum of squares, and $R(\mu,q,g,m,c,r)$ = the total reduction in sum of squares due to fitting all constants. The total reduction in sum of squares is obtained by the same procedures as presented for model (7).

Computing Sums of Squares for the Analysis of Variance

The sums of squares are obtained by the procedures discussed previously, applying the formula

$$B'Z^{-1}B$$

where B' is a row vector of the constant estimates for the effect considered, B is a column vector of these constant estimates, and Z is the segment of the matrix inverse to the variance-covariance matrix corresponding by rows and columns to the particular set of effects. The sums of squares may also be obtained by differences in reductions.

The breakdown of the degrees of freedom for the analysis under model 8 are

Source of Variation	Degrees of Freedom
Total	$n_{...} - 1$
Heterosis	1
Among purebreds	p-1
General combining abilities	p-1
Maternal abilities	p-1
Specific combining abilities	$\frac{p(p-3)}{2}$
Sex-linked effects	$\frac{p(p-3)}{2}+1$
Error	$n_{...} - p^2$

where p is the number of lines included in the experiment.

Construction and Application of Transformation Matrices

The analysis of data of the type described by model (8) can be greatly facilitated by the use of a transformation matrix. The method of constructing a transformation matrix to be presented here for model (8) is a general one and can be followed for other least-squares analyses. As a first step, the least-squares coefficient matrix is written out following the equations described for this model, considering only one observation in each of the p^2 subclasses. Imposing the suggested restrictions on the constant estimates results in the reduced least-squares coefficient matrix presented in Table 21.

Table 21. Reduced least-squares coefficient matrix with one observation per subclass for model

$$Y_{hijk} = \mu + a_h + p_{1ii} + g_{2i} + g_{2j} + m_{2j} + c_{2ij} + r_{2ij} + e_{hijk}$$

$$
\begin{array}{c}
\\ \mu \\ a_1 \\ p_{111} \\ p_{122} \\ p_{133} \\ g_{21} \\ g_{22} \\ g_{23} \\ m_{21} \\ m_{22} \\ m_{23} \\ c_{12} \\ c_{13} \\ r_{12} \\ r_{13} \\ r_{14}
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccccccc}
\mu & a_1 & p_{111} & p_{122} & p_{133} & g_{21} & g_{22} & g_{23} & m_{21} & m_{22} & m_{23} & c_{212} & c_{213} & r_{212} & r_{213} & r_{223}
\end{array}
\\
\begin{bmatrix}
16 & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-8 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 8 & 4 & 4 & 4 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4 & 8 & 4 & 2 & 4 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4 & 4 & 8 & 2 & 2 & 4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4 & 2 & 2 & 6 & 3 & 3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 4 & 2 & 3 & 6 & 3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 2 & 4 & 3 & 3 & 6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 2 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 2 & 6
\end{bmatrix}
\end{array}
$$

This matrix is inverted readily since each of the five small segments can be inverted separately to yield the matrix inverse shown in Table 22. The next step is to write out algebraically the subclasses which are used in measuring each of the effects included in the model. Imposing the appropriate restrictions on these values, the following RHM's are obtained:

$$\mu: \quad \sum_{i}\sum_{j} s_{ij}$$

$$a_1: \quad \sum_{\substack{i\,j \\ i=j}}\sum s_{ij} - \sum_{\substack{i\,j \\ i\neq j}}\sum s_{ij}$$

p_{111}: $s_{11} - s_{44}$

p_{122}: $s_{22} - s_{44}$

p_{133}: $s_{33} - s_{44}$

g_{21}: $(\sum_j s_{1j} + \sum_i s_{i1}) - (\sum_j s_{4j} + \sum_i s_{i4})$

g_{22}: $(\sum_j s_{2j} + \sum_i s_{i2}) - (\sum_j s_{4j} + \sum_i s_{i4})$

g_{23}: $(\sum_j s_{3j} + \sum_i s_{i3}) - (\sum_j s_{4j} + \sum_i s_{i4})$

m_{21}: $\sum_i s_{i1} - \sum_i s_{i4}$

m_{22}: $\sum_i s_{i2} - \sum_i s_{i4}$

m_{23}: $\sum_i s_{i3} - \sum_i s_{i4}$

Table 22. Matrix inverse to least-squares coefficient matrix

$$\begin{array}{rcc}
 & & \begin{array}{cccccccccccccccc} \mu & a_1 & p_{111} & p_{122} & p_{133} & g_{21} & g_{22} & g_{23} & m_{21} & m_{22} & m_{23} & c_{212} & c_{213} & r_{212} & r_{213} & r_{223} \end{array} \\
\begin{array}{r} \mu \\ a_1 \\ p_{111} \\ p_{122} \\ p_{133} \\ g_{21} \\ g_{22} \\ g_{23} \\ m_{21} \\ m_{22} \\ m_{23} \\ c_{12} \\ c_{13} \\ r_{12} \\ r_{13} \\ r_{23} \end{array} & \frac{1}{96} & \left[\begin{array}{cccccccccccccccc}
8 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 72 & -24 & -24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -24 & 72 & -24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -24 & -24 & 72 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 27 & -9 & -9 & -18 & 6 & 6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -9 & 27 & -9 & 6 & -18 & 6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -9 & -9 & 27 & 6 & 6 & -18 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -18 & 6 & 6 & 36 & -12 & -12 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & -18 & 6 & -12 & 36 & -12 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 6 & -18 & -12 & -12 & 36 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & -8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 16 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24 & -12 & +12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -12 & 24 & -12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +12 & -12 & 24
\end{array}\right]
\end{array}$$

c_{212}: $s_{12} + s_{21} - s_{14} - s_{41} - s_{23} - s_{32} + s_{34} + s_{43}$

c_{213}: $s_{13} + s_{31} - s_{14} - s_{41} - s_{23} - s_{32} + s_{24} + s_{42}$

$$r_{212}: s_{12} - s_{14} - s_{42} - s_{21} + s_{24} + s_{41}$$

$$r_{213}: s_{13} - s_{14} - s_{31} + s_{34} + s_{41} - s_{43}$$

$$r_{223}: s_{23} - s_{24} - s_{32} + s_{34} + s_{42} - s_{43}$$

where s_{ij} refers to the subclass for the progeny resulting from a mating of a sire of the i^{th} line with a dam of the j^{th} line.

The third and final step is the multiplication of the RHM's given above by the matrix inverse to the least-squares coefficient matrix shown in Table 22, as follows:

$$\mu: \frac{1}{12} \sum_{ij}\sum s_{ij} + \frac{1}{24} \left(\sum_{\substack{ij \\ i=j}}\sum s_{ij} - \sum_{\substack{ij \\ i\neq j}}\sum s_{ij}\right) = \frac{3}{24} \sum_{\substack{ij \\ i=j}}\sum s_{ij} + \frac{1}{24} \sum_{\substack{ij \\ i\neq j}}\sum s_{ij}$$

$$a_1: \frac{1}{24} \sum_{ij}\sum s_{ij} + \frac{1}{12} \left(\sum_{\substack{ij \\ i=j}}\sum s_{ij} - \sum_{\substack{ij \\ i\neq j}}\sum s_{ij}\right) = \frac{3}{24} \sum_{\substack{ij \\ i=j}}\sum s_{ij} - \frac{1}{24} \sum_{\substack{ij \\ i\neq j}}\sum s_{ij}$$

$$p_{111}: \frac{3}{4} (s_{11}-s_{44}) - \frac{1}{4} (s_{22}-s_{44}) - \frac{1}{4} (s_{33}-s_{44}) = \frac{3}{4} s_{11} - \frac{1}{4} s_{22} - \frac{1}{4} s_{33} - \frac{1}{4} s_{44}$$

$$\vdots$$

$$r_{223}: \frac{1}{8} (s_{12}-s_{14}-s_{42}-s_{21}+s_{24}+s_{41}) - \frac{1}{8} (s_{13}-s_{14}-s_{31}+s_{34}+s_{41}-s_{43})$$

$$+ \frac{1}{4} (s_{23}-s_{24}-s_{32}+s_{34}+s_{42}-s_{43}) = \frac{1}{8} s_{12} - \frac{1}{8} s_{13} - \frac{1}{8} s_{21} + \frac{1}{4} s_{23} - \frac{1}{8} s_{24}$$

$$+ \frac{1}{8} s_{31} - \frac{1}{4} s_{32} + \frac{1}{8} s_{34} + \frac{1}{8} s_{42} - \frac{1}{8} s_{43}$$

The values resulting from the above multiplications are the coefficients for the transformation matrix, K, which is shown in Table 23. If D is the least-squares coefficient matrix for the subclasses, then the matrix inverse, C^{-1}, to the least-square coefficient matrix for model (8) is obtained by

$$KD^{-1} K'$$

where K is the transformation matrix, D^{-1} the matrix inverse of D, and K' is the transpose of the transformation matrix. Since D is a diagonal matrix, its inverse can be obtained quickly by calculating the reciprocal of each diagonal element. The constant estimates are computed by

$$KS = B$$

where S is a column vector of the subclass means and B is the column vector of the constant estimates.

Table 23. Transformation matrix for model. $y_{hijk}=\mu+a_h+p_{1ii}+g_{2i}+g_{2j}+m_{2j}+c_{2ij}+r_{2ij}+e_{hijk}$. Purebred effects estimated independently of general combining ability and maternal ability effects.

		s_{11}	s_{12}	s_{13}	s_{14}	s_{21}	s_{22}	s_{23}	s_{24}	s_{31}	s_{32}	s_{33}	s_{34}	s_{41}	s_{42}	s_{43}	s_{44}
μ:		3	1	1	1	1	3	1	1	1	1	3	1	1	1	1	3
a_1:		3	-1	-1	-1	-1	3	-1	-1	-1	-1	3	-1	-1	-1	-1	3
p_{111}:		18	0	0	0	0	-6	0	0	0	0	-6	0	0	0	0	-6
p_{122}:		-6	0	0	0	0	18	0	0	0	0	-6	0	0	0	0	-6
p_{133}:		-6	0	0	0	0	-6	0	0	0	0	18	0	0	0	0	-6
g_{21}:		0	6	6	6	0	0	-3	-3	0	-3	0	-3	0	-3	-3	0
g_{22}:	$\frac{1}{24}$	0	0	-3	-3	6	0	6	6	-3	0	0	-3	-3	0	-3	0
g_{23}:		0	-3	0	-3	-3	0	0	-3	6	6	0	6	-3	-3	0	0
m_{21}:		0	-6	-6	-6	6	0	0	0	6	0	0	0	6	0	0	0
m_{22}:		0	6	0	0	-6	0	-6	-6	0	6	0	0	0	6	0	0
m_{23}:		0	0	6	0	0	0	6	0	-6	-6	0	-6	0	0	6	0
c_{212}:		0	4	-2	-2	4	0	-2	-2	-2	-2	0	4	-2	-2	4	0
c_{213}:		0	-2	4	-2	-2	0	-2	4	4	-2	0	-2	-2	4	-2	0
r_{212}:		0	6	-3	-3	-6	0	3	3	3	-3	0	0	3	-3	0	0
r_{213}:		0	-3	6	-3	3	0	-3	0	-6	3	0	3	3	0	-3	0
r_{223}:		0	3	-3	0	-3	0	6	-3	3	-6	0	3	0	3	-3	0

It can be seen that if a simple transformation matrix is available for the model desired, a great deal of labor can be saved in this type of analysis. While many transformation matrices can be written readily without using the procedure outlined above, those for more complex designs can be obtained in this manner. As had been pointed out previously, transformation matrices are also useful when making orthogonal comparisons among the constant estimates of treatment effects, where one is concerned with only a segment of the entire matrix inverse, and only those estimates of the constants for the treatment effects under consideration.

Numerical Example

The data used in the numerical example are those used previously for model (7), with the addition of the linebreds, which were included in the experiment but omitted for the previous example.

Least-Squares Analysis

The least squares equations are shown in Table 24 where it can be seen that only the segment corresponding by row and column to the general combining ability effects contains entries in the off-diagonals. All other segments corresponding by row and column to one of the effects being estimated

Table 24. Least-squares equations.

	μ	a_1	a_2	p_{111}	p_{122}	p_{133}	p_{144}	g_{21}	g_{22}	g_{23}	g_{24}	m_{21}	m_{22}	m_{23}	m_{24}	c_{212}	c_{213}	c_{214}	c_{223}	c_{224}	c_{234}	r_{212}	r_{213}	r_{214}	r_{221}	r_{223}	r_{224}	r_{231}	r_{232}	r_{234}	r_{241}	r_{242}	r_{243}	RHM
μ	208	72	136	22	26	12	12	72	64	68	68	38	32	32	34	24	22	26	22	18	24	12	10	12	12	10	10	12	12	12	14	8	12	12699.2
a_1	72	72		22	26	12	12																											4333.1
a_2	136		136					72	64	68	68	38	32	32	34	24	22	26	22	18	24	12	10	12	12	10	10	12	12	12	14	8	12	8366.1
p_{111}	22	22		22																														1231.0
p_{122}	26	26			26																													1725.0
p_{133}	12	12				12																												645.5
p_{144}	12	12					12																											731.6
g_{21}	72		72					72	24	22	26	38	12	10	12	24	22	26				12	10	12	12			12			14			4519.8
g_{22}	64		64					24	64	22	18	12	32	10	10	24			22	18		12			12	10	10		12			8		3920.2
g_{23}	68		68					22	22	68	24	12	12	32	12		22		22		24		10			10		12	12	12			12	4168.9
g_{24}	68		68					26	18	24	68	14	8	12	34			26		18	24			12			10			12	14	8	12	4123.3
m_{21}	38		38					38	12	12	14	38				12	12	14							12			12			14			2344.1
m_{22}	32		32					12	32	12	8		32			12			12	8		12							12			8		2000.3
m_{23}	32		32					10	10	32	12			32			10		10		12		10			10							12	1323.8
m_{24}	34		34					12	10	12	34				34			12		10	12			12			10			12				2197.9
c_{212}	24		24					24	24			12	12			24						12			12									1479.2
c_{213}	22		22					22		22		12		10			22						10					12						1392.3
c_{213}	26		26					26			26	14			12			26						12							14			1648.3
c_{223}	22		22						22	22			12	10					22							10			12					1371.3
c_{224}	18		18						18		18		8		10					18							10					8		1069.7
c_{234}	24		24							24	24			12	12						24									12			12	1405.3
r_{212}	12		12					12	12				12			12						12												767.2
r_{213}	10		10					10		10				10			10						10											621.0
r_{214}	12		12					12			12				12			12						12										787.5
r_{221}	12		12					12	12			12				12									12									712.0
r_{223}	10		10						10	10				10					10							10								597.4
r_{224}	10		10						10		10				10					10							10							610.5
r_{231}	12		12					12		12		12					12											12						771.3
r_{232}	12		12						12	12			12						12										12					773.9
r_{234}	12		12							12	12				12						12									12				799.9
r_{241}	14		14					14			14	14						14													14			860.8
r_{242}	8		8						8		8		8							8												8		459.2
r_{243}	12		12							12	12			12							12												12	605.4

contain entries in the diagonals only. It can also be observed that the sum of the coefficients for the general combining ability effects (in all equations in which there are entries) equals twice the coefficient for μ, except in the μ equation due to inclusion of purebreds in the analysis.

After imposing the restrictions on the constant estimates suggested for this model, the coefficient matrix can be inverted and the inverse elements necessary for estimating constants, standard errors, and calculating sums of squares for the analysis of variance can be obtained. The constants may be estimated during the process of inversion.

When a transformation matrix is available the inverse elements and the constant estimates can be obtained more readily than by the inversion of the least-squares coefficient matrix. Using the transformation matrix which has been presented for this model, the desired matrix inverse to the least-squares coefficient matrix can be calculated by means of a desk calculator in a very short time. The constants estimates are also computed simply and quickly.

Since the same observations were utilized in estimating the general combining ability, maternal, specific combining ability, and sex-linked effects as in the previous model, the inverse elements pertaining to these effects and the constant estimates are the same as were obtained in the previous analysis, within rounding error. With the inclusion of the linebreds in this analysis, additional constant estimates are obtained for heterotic and purebred effects. The constant estimates resulting from the analysis under model (3) are shown below:

$\hat{\mu} = 60.322$	$\hat{g}_{22} = -1.293$	$\hat{c}_{13} = -.172$
$\hat{a}_1 = -1.057$	$\hat{g}_{23} = 2.753$	$\hat{r}_{12} = .766$
$\hat{p}_{111} = -3.310$	$\hat{m}_{21} = -1.641$	$\hat{r}_{13} = .982$
$\hat{p}_{122} = 7.081$	$\hat{m}_{22} = 1.426$	$\hat{r}_{23} = 1.229$
$\hat{p}_{133} = -5.473$	$\hat{m}_{23} = -5.784$	
$\hat{g}_{21} = 2.941$	$\hat{c}_{12} = -1.285$	

The remaining constant estimates ($\hat{p}_{144}$, $\hat{g}_{24}$, etc.) can be obtained by considering the restrictions that were imposed on the estimates and making the appropriate additions and subtractions. Due to the size of the matrix inverse, this is not presented.

The total reduction in sums of squares is calculated by the usual procedures of obtaining the sum of the products of the constant estimates by the corresponding RHM's

$$R(\mu,a,p,g,m,c,r) = 779{,}783.44$$

This value can also be obtained by calculating the uncorrected sums of squares for the subclasses.

The error sum of squares is then calculated as the difference between the total uncorrected sum of squares and the reduction due to fitting all

constants,

$$\text{Error sum of squares} = 793{,}111.89 - 779{,}783.44 = 13{,}328.45.$$

The sums of squares for heterosis, purebreds, general combining ability, maternal, specific combining ability, and sex-linked effects are obtained by the procedures previously outlined. Completion of these sums of squares results in the following analysis of variance:

Source of Variation	d.f.	Sums of Squares	Mean Squares
Heterosis (A)	1	192.93	192.93
Purebreds (P)	3	1878.50	626.17
General combining abilities(G)	3	1041.44	347.15
Maternal abilities (M)	3	1725.26	575.09
Specific combining abilities(C)	2	180.73	90.36
Sex-linked effects (R)	3	267.73	89.24
Error	192	13328.45	69.42

In view of the small sum of squares for the sex-linked effects, these effects could be eliminated from the model and new estimates computed for the remaining effects. Should the specific combining ability effects prove to be non-significant after the new estimates have been made, these effects could also be eliminated from the model and more efficient estimates made of the remaining effects.

If standard errors are desired, they can be computed from the formulas previously presented, as follows:

$$s_{\hat{\mu}} = \sqrt{.00578994}\ \sqrt{69.42} = (.0761)(8.33) = .63$$

$$s_{(\hat{\mu}+\hat{a}_1)} = \sqrt{.00578994 + .00578994 + 2(.00204077)}\ (8.33) = (.125)(8.33)$$
$$= 1.04$$

$$s_{(\hat{\mu}+\hat{a}_1+\hat{p}_{111})} = \left[.00578994 + .00578994 + .03838870 + 2(.00204077) + 2(-.00214889) + 2(-.00214889)\right]^{\frac{1}{2}} (8.33) = (.213)(8.33)$$
$$= 1.77$$

The standard error for a linecross or crossbred group can most easily be obtained from a consideration of the diagonal matrix made up of the subclass (linebred and crossbred groups) numbers. Each of these subclasses contain the $\hat{\mu}$ effect plus subclass effect, so the inverse element from this diagonal element corresponding to a subclass, $\hat{s}_{ij}$, is the element needed to determine the standard error, $s_{\hat{\mu}+\hat{s}_{ij}}$. To calculate the standard error of the cross between a sire of the 1st line and a dam of the 2nd line, the subclass number is 12 and its inverse 1/12. Therefore, the standard error for this crossbred or linebred group is

$$s_{12} = \sqrt{\frac{1}{12}}\ \sqrt{69.42} = \sqrt{5.7850} = 2.41$$

The data which have been used in this analysis could also have been analyzed using the model

$$y_{ijk} = \mu + s_i + d_j + (sd)_{ij} + e_{ijk} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where y_{ijk} = the observation on the k^{th} offspring of a mating between the j^{th} dam and i^{th} sire. While an analysis of this model does not yield the detailed information of the previous analysis, it does offer a measure of the over-all performance of the lines as sires and dams. The equations for an analysis under this model would be written in the usual way for a two-way classification with interaction. Again, the availability of a transformation matrix lessens the labor considerably. The transformation matrix for model (9) is shown in Table 25 for the case of four lines.

Table 25. Transformation matrix for model $y_{ijk} = \mu+s_i+d_j+(sd)_{ij}+e_{ijk}$.

		s_{11}	s_{12}	s_{13}	s_{14}	s_{21}	s_{22}	s_{23}	s_{24}	s_{31}	s_{32}	s_{33}	s_{34}	s_{41}	s_{42}	s_{43}	s_{44}
μ		1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
s_1		3	3	3	3	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
s_2		-1	-1	-1	-1	3	3	3	3	-1	-1	-1	-1	-1	-1	-1	-1
s_3		-1	-1	-1	-1	-1	-1	-1	-1	3	3	3	3	-1	-1	-1	-1
d_1		3	-1	-1	-1	3	-1	-1	-1	3	-1	-1	-1	3	-1	-1	-1
d_2		-1	3	-1	-1	-1	3	-1	-1	-1	3	-1	-1	-1	3	-1	-1
d_3		-1	-1	3	-1	-1	-1	3	-1	-1	-1	3	-1	-1	-1	3	-1
sd_{11}	$\frac{1}{16}$	9	-3	-3	-3	-3	1	1	1	-3	1	1	1	-3	1	1	1
sd_{12}		-3	9	-3	-3	1	-3	1	1	1	-3	1	1	1	-3	1	1
sd_{13}		-3	-3	9	-3	1	1	-3	1	1	1	-3	1	1	1	-3	1
sd_{21}		-3	1	1	1	9	-3	-3	-3	-3	1	1	1	-3	1	1	1
sd_{22}		1	-3	1	1	-3	9	-3	-3	1	-3	1	1	1	-3	1	1
sd_{23}		1	1	-3	1	-3	-3	9	-3	1	1	-3	1	1	1	-3	1
sd_{31}		-3	1	1	1	-3	1	1	1	9	-3	-3	-3	-3	1	1	1
sd_{32}		1	-3	1	1	1	-3	1	1	-3	9	-3	-3	1	-3	1	1
sd_{33}		1	1	-3	1	1	1	-3	1	-3	-3	9	-3	1	1	-3	1

An analysis of the data of the present example using model (9) yielded the following constant estimates:

$\hat{\mu} = 60.850$	$\hat{d}_3 = -4.330$	$(\hat{sd})_{23} = 2.452$
$\hat{s}_1 = 1.053$	$(\hat{sd})_{11} = -5.361$	$(\hat{sd})_{31} = 2.559$
$\hat{s}_2 = .767$	$(\hat{sd})_{12} = -.163$	$(\hat{sd})_{32} = -.005$
$\hat{s}_3 = 1.454$	$(\hat{sd})_{13} = 4.526$	$(\hat{sd})_{33} = -4.183$
$\hat{d}_1 = -.588$	$(\hat{sd})_{21} = -1.696$	
$\hat{d}_2 = 2.193$	$(\hat{sd})_{22} = 2.536$	

The remaining constant estimates for sires, dams, and interactions can be obtained by the usual procedures. Total reduction in sum of squares, error sum

of squares, and the sums of squares for sires, dams, and interactions also follow the methods previously outlined. The analysis of variance following the completion of these computations is shown below:

Source of Variation	d.f.	Sums of Squares	Mean Squares
Sires (S)	3	658.49	219.50
Dams (D)	3	1409.62	469.87
S x D	9	1812.72	201.41
Error	191	13327.55	69.78

The significant interaction of sires and dams is not unexpected in an experiment of this type and makes the analysis under model (9) of very limited value. If one is concerned only with the particular crosses produced in the experiment, this analysis does measure the value of the lines as sires and dams. However, there is no information as to what type of gene action is involved and so the analysis is not of value in formulating breeding plans where one wants to have information on both additive and non-additive effects, such as are brought out under models (7) or (8). Since the interaction is significant it is of importance to know what contribution specific combining ability and sex-linked effects have made.

Analysis of the data under model (8) yielded an estimate of the over-all amount of heterosis and a test of the significance of this effect. The use of a simple transformation matrix makes it possible to obtain estimates of the amount of heterosis and the inverse elements necessary to test the significance of the heterosis effect in each cross, and also to make tests of significance among the heterosis estimates. While the constant estimates and the tests of significance are not independent, since the purebreds or linebreds are each used three times, they are of considerable interest. The transformation matrix is shown in Table 26 together with the transformed matrix inverse of the subclass numbers. It can be seen from the transformation matrix that the constant estimate for heterosis in each cross is a contrast between the average of the purebreds and the average of the reciprocal crosses. The ranked constant estimates, with tests of the significance of each estimate and tests of the significance among the estimates, are shown below:

Cross	Constant Estimate	Tests of Significance Among Crosses
13	8.314**	
14	5.095*	
23	2.047	
34	1.175	
12	.483	
24	-4.431	

The tests of significance of the heterosis estimate for each cross was made by the analysis of variance, with a single degree of freedom for each estimate. The sum of squares for each contrast utilizes the constant estimate and the inverse element of the transformed matrix corresponding by row and column to the cross being considered. For example, the sum of squares for the estimate of heterosis resulting from crossing line 1 and line 3 is

$$\text{S.Sqs.---}h_{13} = \frac{(8.314)^2}{.07803030} = 885.84 \quad .$$

Table 26. Transformation matrix used to obtain constant estimates of heterosis effects for separate crosses and the transformed matrix inverse of subclass numbers.

	s_{11}	s_{12}	s_{13}	s_{14}	s_{21}	s_{22}	s_{23}	s_{24}	s_{31}	s_{32}	s_{33}	s_{34}	s_{41}	s_{42}	s_{43}	s_{44}
h_{12}:	-.5	.5	0	0	.5	-.5	0	0	0	0	0	0	0	0	0	0
h_{13}:	-.5	0	.5	0	0	0	0	0	.5	0	-.5	0	0	0	0	0
h_{14}:	-.5	0	0	.5	0	0	0	0	0	0	0	0	.5	0	0	-.5
h_{23}:	0	0	0	0	0	-.5	.5	0	0	.5	-.5	0	0	0	0	0
h_{24}:	0	0	0	0	0	-.5	0	.5	0	0	0	0	0	.5	0	-.5
h_{34}:	0	0	0	0	0	0	0	0	0	0	-.5	.5	0	0	.5	-.5

	h_{12}	h_{13}	h_{14}	h_{23}	h_{24}	h_{23}
h_{12}	.06264569	.01136364	.01136364	.00961538	.00961538	.00000000
h_{13}	.01136364	.07803030	.01136364	.02083333	.00000000	.02083333
h_{14}	.01136364	.01136364	.07087744	.00000000	.02083333	.02083333
h_{23}	.00961538	.02083333	.00000000	.07628205	.00961538	.02083333
h_{24}	.00961538	.00000000	.02083333	.00961538	.08669872	.02083333
h_{34}	.00000000	.02083333	.02083333	.02083333	.02083333	.08333333

The analysis of variance is shown below:

Source of Variation	d.f.	Sum of Squares	Mean Square
Heterosis			
Line 1 and line 2	1	3.72	3.72
Line 1 and line 3	1	885.93	885.93**
Line 1 and line 4	1	366.16	366.16*
Line 2 and line 3	1	54.93	54.93
Line 2 and line 4	1	226.51	226.51
Line 3 and line 4	1	16.57	16.57
Error	191	13327.55	69.78

The tests of significance among the constant estimates for the separate crosses were made by means of Duncan's Multiple Range Test, the details of which have been presented previously.

SIMULTANEOUS ANALYSIS OF DATA FROM CROSSES AND PUREBREDS II

The analysis of the data under model (8) estimated the purebred effects independently of the general combining ability effects. The purebred effects can also be estimated by a simultaneous consideration of the general combining ability and maternal effects by including these effects in the

equations for the purebred effects. This method of analysis results in the same constant estimates and sums of squares for all effects other than the purebreds.

Model

The mathematical model describing this third method of analysis is:

$$y_{hijk} = \mu + a_h + p_{1ii} + g_{.i} + g_{.j} + m_{.j} + c_{2ij} + r_{2ij} + e_{hijk} \quad . \quad . \quad . \quad . \quad (10)$$

where the symbols represent the same effects as those for model (8). The subscripts indicate a simultaneous consideration of the general combining ability, maternal, and purebred effects. The least-squares equations for model (10) are presented in Table 27.

Completion of Analysis

The restrictions imposed on the constant estimates are the same as those for model (8) and the remainder of the calculations proceed by the same methods as outlined for models (7) and (8). No numerical example is presented for this model due to its similarity to model (8).

Table 27. Least-squares Equations for Model (10).

	$\hat{\mu}$	$\hat{a}_h$	$\hat{p}_{1ii}$	$\hat{g}_{.i}$	$\hat{m}_{.j}$	$\hat{c}_{2ij}$	$\hat{r}_{2ij}$	RHM
μ	$n_{...}$	$n_{h..}$	n_{1ii}	$n_{.i.}+n_{..i}$	$n_{..j}$	$n_{2ij}+n_{2ji}$	n_{2ij}	$Y_{...}$
a_h	$n_{h..}$	${}_0n_{h..}^{0}$	n_{1ii}	$n_{hi.}+n_{h.i}$	$n_{h.j}$	$n_{2ij}+n_{2ji}$	n_{2ij}	$Y_{h..}$
p_{1ii}	n_{1ii}	n_{1ii}	${}_0n_{1ii}^{0}$	n_{1ii}	n_{1ii}	—	—	Y_{1ii}
$g_{.i.}$	$n_{.i.}+n_{..i}$	$n_{hi.}+n_{h.i}$	n_{1ii}	$n_{2ij}+n_{2ji}$ $n_{.i.}+n_{..i}$ $n_{2ij}+n_{2ji}$	$n_{..j}$	$n_{2ij}+n_{2ji}$	n_{2ij}	$Y_{.i.}+Y_{..i}$
$m_{.j}$	$n_{..j}$	$n_{h.j}$	n_{1ii}	$n_{..j}$	${}_0n_{..j}^{0}$	$n_{2.j}$	n_{2ij}	$Y_{..j}$
c_{2ij}	$n_{2ij}+n_{2ji}$	$n_{2ij}+n_{2ji}$	—	$n_{2ij}+n_{2ji}$	$n_{2.j}$	${}_0n_{2ij}+n_{2ji}^{0}$	n_{2ij}	$Y_{2ij}+Y_{2ji}$
r_{2ij}	n_{2ij}	n_{2ij}	—	n_{2ij}	n_{2ij}	n_{2ij}	${}_0n_{2ij}^{0}$	Y_{2ij}